ELECTRICAL ENGINEERING

THE THEORY AND CHARACTERISTICS OF ELECTRICAL CIRCUITS AND MACHINERY

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PREFACE

THIS book has been compiled as a foundation for lecture courses junior and senior students in Electrical Engineering.

The theory and characteristics of electrical machines are developed from the fundamental principles of electrostatics and electromagnetics. Only the more standard types have been discussed since familiarity with the principles of their operations will guide the student to a complete understanding of other machines which differ only in minor respects. This general groundwork may be extended to suit the requirements of particular classes.

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ELECTRICAL ENGINEERING

CHAPTER I

ELECTROSTATICS

1. Electrification. Bodies which are charged with electricity are said to be electrified. Charges are of two kinds called positive and negative. Bodies which have a positive charge are acted upon by forces tending to make them give up their charge; bodies which have a negative charge are acted upon by forces tending to convey a positive charge to them equal to their negative charge. These forces are exerted through the medium separating the charges and the medium is in a state of stress.

The body with the positive charge is at a higher potential than the body with the negative charge and the difference of potential between the two is a measure of the tendency for electricity to pass from one to the other.

2. Electrical Conductors and Insulators. If two metallic bodies charged to different potentials are joined by a metal wire, electricity will flow from one to the other until the potential of both is the same and the transfer of electricity will take place almost instantaneously. The metal wire is therefore a good conductor of electricity; or, it offers a low resistance to the passage of electricity through it.

If the two charged bodies had been joined by a glass rod, there would have been no transfer of electricity between them, or, it would have taken place so slowly that it could only be detected by the most delicate instruments. Glass is therefore a very bad conductor; or, it offers a very high resistance to the passage of electricity. It is called a non-conductor or insulator.

As all materials conduct to a certain extent, it is not possible to divide them absolutely into conductors and insulators, but, since the resistance of a good insulator is many million times that of a good conductor, they may be so divided for practical purposes.

In the first class are silver, which is the best conductor, copper and other metals, graphite, impure water and solutions of salts. In the second class are air, which when dry is an almost perfect insulator, glass, paraffin, ebonite, porcelain, rubber, shellac, oils and the numerous insulating compounds used in electrical engineering.

3. Electrostatics and Electromagnetics. Electrostatics comprises phenomena related to electric charges at rest and to the stresses produced in the field surrounding them. These phenomena become of great importance where very large differences of potential must be provided for, as for example in the design and operation of all high voltage apparatus and systems.

Electromagnetics comprises phenomena related to electricity in motion, that is, to currents of electricity and the magnetic fields produced by them. Almost all the problems to be solved by the electrical engineer come under this head.

4. Laws of Electrostatics. *First Law.* Like charges of electricity repel one another; unlike charges attract one another.

Second Law. The force exerted between two charges of electricity is proportional to the product of their strengths and is inversely proportional to the square of the distance between them; it also depends on the nature of the medium separating them.

This law can be expressed by the formula,

$$f = \frac{qq_1}{Kr^2}, \qquad \qquad (1)$$

where q and q_1 are the charges of electricity, r is the distance between them in centimeters, K is a constant depending on the medium separating the charges and is called its specific inductive capacity or dielectric constant. The unit of quantity is so chosen that the dielectric constant for air is unity; for all other substances it is greater than unity. f is the force in dynes exerted between the two charges; if the charges are of the same kind the force between them is a repulsion and f is positive.

One electrostatic unit of quantity is that quantity which, when placed at a distance of one centimeter in air from a similar quantity, repels it with a force of one dyne.

The practical unit of quantity is the coulomb; one coulomb is 3×10^9 electrostatic units.

5. Dielectric Constants. The following table gives the dielectric constants or specific inductive capacities of some of the most common dielectrics.

Material	Dielectric Constant
Air	1.0
Vacuum	0.9994
Transformer oil	2.1
Shellac	2.75
Paraffin wax	2.3
Rubber	2.35
Gutta percha	3.0–5.0
Ebonite	2.8
Glass	5.0–10.0
Mica	5.0–7.0
Conductors	infinity

Dielectric constants generally decrease slightly with increasing temperature and in some cases they depend on the intensity of the electrostatic field.

6. Electrostatic Field. Any space in which electrostatic forces act is called an electrostatic field. The direction of the force at any point in the field is the direction in which a unit positive charge placed at the point tends to move and its intensity is the force in dynes exerted on the unit charge.

The electrostatic field is conveniently represented by lines of electrostatic induction or dielectric flux drawn in the direction of the force. In air the number of lines per square centimeter is equal to the force in dynes at the point and in a medium of dielectric constant K the number of lines per square centimeter is equal to K times the force. This may be stated in another way: Unit electrostatic force produces one line of dielectric flux per square centimeter in air and K lines per square centimeter in a medium of dielectric constant K.

The electrostatic force at a point is expressed in dynes and is represented by $\mathfrak{F}$; the dielectric flux density at a point is expressed in lines per square centimeter and is represented by $\mathfrak{D}$.

7. Field Surrounding a Point Charge. At a distance r cm. in air from an isolated charge q, a unit charge is acted on by a force

$$\mathfrak{F} = \frac{q}{r^2} \text{ dynes}; \qquad (2)$$

and the dielectric flux density produced at the point is

$$\mathfrak{D} = \mathfrak{F} = \frac{q}{r^2} \text{ lines per square centimeter} \qquad (3)$$

This density is produced over the surface of a sphere of radius r and therefore the total dielectric flux from the charge q is

$$\psi = \frac{q}{r^2} \times 4\,\pi r^2 = 4\,\pi q \text{ lines.}$$

Consider the same charge surrounded by a medium of dielectric constant K.

The force exerted on a unit charge at a distance r cm. from q is

$$\mathfrak{F} = \frac{q}{Kr^2} \text{ dynes;}$$

the dielectric flux density produced is

$$\mathfrak{D} = \mathfrak{F}K = \frac{q}{r^2} \text{ lines per square centimeter,}$$

and therefore the dielectric flux from charge q is, as before,

$$\psi = 4\,\pi q \text{ lines.} \qquad (4)$$

Fig. 1 represents the electrostatic field about a positive point charge. The lines of flux extend out radially in all directions.

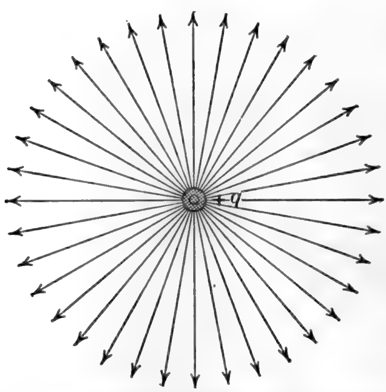


FIG. 1. Field surrounding a point charge.

8. Dielectric Flux from a Unit Charge. The dielectric flux from a unit charge is $4\,\pi$ lines by equation 4. Thus if a dielectric flux ψ starts from any surface the positive charge on that surface is $q = \frac{\psi}{4\,\pi}$ units, and if a flux ψ ends on any surface the negative charge on that surface is $-\,q = \frac{\psi}{4\,\pi}$ units.

A unit positive charge is always associated with each 4π lines leaving a surface and a unit negative charge with each 4π lines entering a surface.

9. Field between Two Point Charges. The field between two point charges q and $-q$ consists of curved lines extending from the positive to the negative charge.

The direction and intensity of the force at any point may be

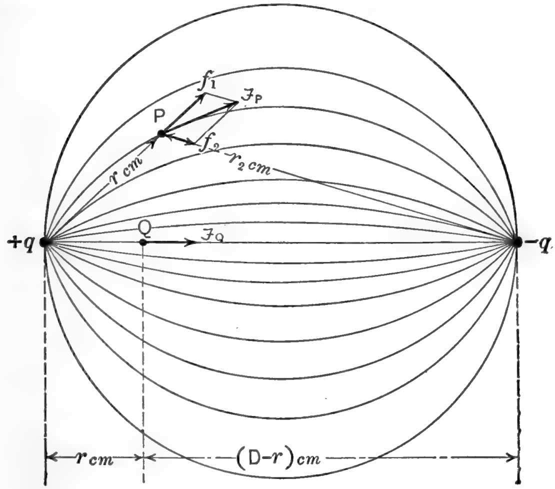


FIG. 2. Field between two point charges.

obtained as shown in Fig. 2. If a unit positive charge is placed at any point P it is repelled by q with a force $f_1 = \frac{q}{r_1^2}$ and is attracted by $-q$ with a force $f_1 = \frac{q}{r_2^2}$. The resultant force at the point is $\mathfrak{F}$ and it is the vector sum of f_1 and f_2. If D cm. is the distance between the charges, the force at a point on the line joining the charges distant r cm. from the charge q is

$$\mathfrak{F}_Q = \frac{q}{r^2} + \frac{q}{(D-r)^2} \text{ dynes.}$$

Since the medium is air the flux density at any point is numerically equal to the force $\mathfrak{F}$ at the point.

10. Field between Parallel Plates. The field between two parallel plates A and B, Fig. 3, charged respectively with $+q$ and $-q$ units per square centimeter consists of straight lines

normal to the plates except near the edges where they become slightly curved.

The dielectric flux density is uniform throughout the volume between the plates, and its value is

$$\mathfrak{D} = 4\,\pi q \text{ lines per square centimeter.} \qquad (5)$$

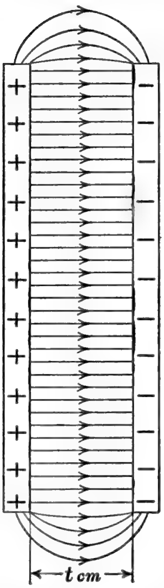


FIG. 3. Field between parallel plates.

If the medium between the plates is air, the electrostatic force at any point is

$$\mathfrak{F} = \mathfrak{D} = 4\,\pi q \text{ dynes.} \quad . \quad . \quad . \quad (6)$$

If the medium has a dielectric constant K, the force is

$$\mathfrak{F} = \frac{\mathfrak{D}}{K} = \frac{4\,\pi q}{K} \text{dynes.} \quad . \quad . \quad . \quad (7)$$

11. Potential. The difference of potential between two points is measured by the work done in carrying a unit charge from one to the other against the electrostatic force in the field; it is therefore the line integral of the force between the points.

The difference of potential between the two plates in Fig. 3 is

$$e = \int_0^t \mathfrak{F}\, dr, \quad . \quad . \quad . \quad . \quad (8)$$

where t cm. is the distance between the plates.

Difference of potential tends to cause electricity to flow from one point to the other and is therefore called electromotive force.

Unit difference of potential (electrostatic) exists between two points when one erg of work is done in conveying unit charge from one to the other.

The practical unit difference of potential or electromotive force is called the volt. One electrostatic unit is equal to 300 volts.

The earth is usually taken as the arbitrary zero of potential, and the potentials of other bodies are given with reference to it.

If a difference of potential is produced between two points on a conductor, electricity will flow from the point of high potential until the potentials of all parts of the conductor are the same.

When, however, the difference of the potential is maintained by an electric generator a current of electricity flows continuously from one point to the other.

When a difference of potential is produced between two conductors separated by an insulating material, the material is subjected to a stress and lines of dielectric flux pass through it.

12. Induced Charges of Electricity. When a positively charged body A, Fig. 4, is brought near to an insulated conductor

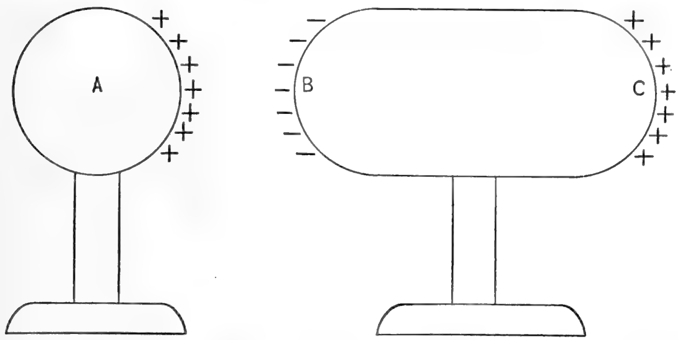


FIG. 4. Induced charges.

BC, a negative charge is induced on the nearer side B and an equal positive charge on the farther side C. The explanation is that the potential of the end B due to the charge on A is higher

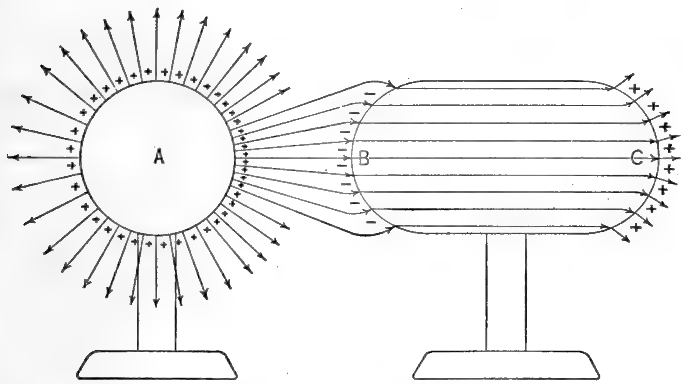


FIG. 5. Induced charges.

than that of C, but, since BC is a conductor, as soon as there is a difference of potential between two points on it, electricity flows from the region of high potential at B to the region of low potential at C; thus a positive charge appears at C and an equal negative charge at B and these charges so distribute themselves over

the surface of BC that the potential at every point on it is the same, being the sum of the potentials due to the charge on A and the two charges on BC.

The same result is illustrated in Fig. 5, which shows the dielectric flux produced.

The flux from A extends out radially in all directions. In the region surrounding BC the lines are deflected and a large number pass through the conductor BC, since its dielectric constant is infinity and therefore it offers an easy path. For every 4π lines entering at B a unit negative charge appears on the surface, and for every 4π lines leaving at C a unit positive charge appears.

13. Equivalent Charges. A uniformly distributed charge on the surface of a sphere acts as though it were concentrated at the centre. If a sphere of radius R cm. has a charge Q uniformly distributed over its surface, the density of the charge is $\frac{Q}{4\pi R^2}$; and since 4π lines emanate from unit charge, the flux density at the surface of the sphere is $\frac{4\pi Q}{4\pi R^2} = \frac{Q}{R^2}$ lines per square centimeter. If the charge Q is concentrated at the centre of the sphere, then, by formula 3, the flux density at a distance R cm. from the charge is $\frac{Q}{R^2}$ and therefore the uniformly distributed charge on the surface of the sphere may be represented by an equal charge concentrated at the centre.

Similarly a uniformly distributed charge on the surface of a cylinder may be represented by an equal charge uniformly distributed along the axis of the cylinder.

14. Distribution of Potential in the Space Surrounding a Point Charge. In Fig. 6 a positive charge q is placed at the point O at an infinite distance from all other charges. The potential at a point P_1, distant r_1 cm. from O, is the work done in moving unit charge from a point of zero potential to the point P_1 against the forces in the field. The intensity of the force at a distance r cm. from O is by formula

$$\mathfrak{F} = \frac{q}{r^2} \text{ dynes};$$

the work done in moving a unit charge against this force through

a distance dr is

$$\mathfrak{F}\, dr = q \frac{dr}{r^2} \text{ ergs};$$

and the work done in moving the charge from a point of zero potential to P_1 is

$$\begin{aligned} e_1 &= \int_{r_1}^{\infty} \mathfrak{F}\, dr = \int_{r_1}^{\infty} q \frac{dr}{r^2} \\ &= q\left[-\frac{1}{r}\right]_{r_1}^{\infty} \\ &= \frac{q}{r_1} \text{ ergs}; \end{aligned} \quad \ldots\ldots\ldots \quad (9)$$

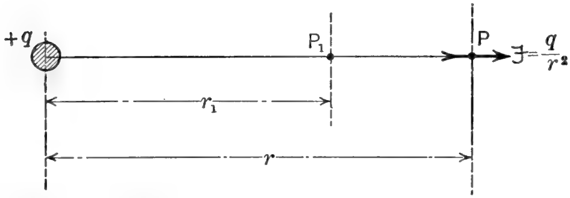


FIG. 6. Distribution of potential about a point charge.

therefore, the potential at a point due to a charge of electricity is equal to the charge in electrostatic units divided by its distance in centimeters from the point.

15. Potential at a Point Due to a Number of Charges. If

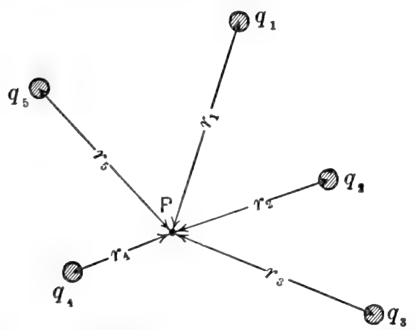


FIG. 7. Potential due to a number of charges.

there are several charges q_1, q_2, q_3, etc. in the field, the potential at any point is the sum of the potentials due to the separate charges and is

$$e = \frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \text{etc.} = \sum \frac{q}{r}, \quad \ldots\ldots \quad (10)$$

where r_1, r_2, r_3, etc. are the distances from the various charges to the point. Fig. 7.

Potential at a Point Due to a Charged Surface. In Fig. 8 AB is a surface with a non-uniform distribution of charge over it;

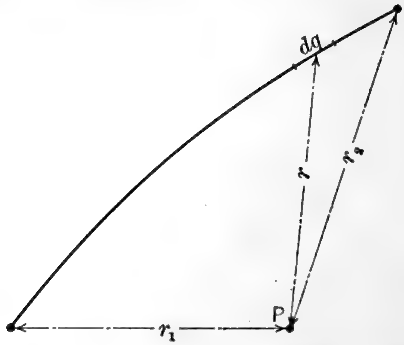


FIG. 8. Potential due to a charged surface.

if dq is a small element of charge at a distance r cm. from the point P, the potential at P due to the charge dq is

$$de = \frac{dq}{r},$$

and the potential due to the total charge on the surface is

$$e = \int \frac{dq}{r} \qquad (11)$$

16. Equipotential Surfaces. Surfaces of which all points are at the same potential are called equipotential surfaces.

In Fig. 9 A is an isolated sphere of radius R, charged with Q units of electricity. Any spherical surface drawn about the centre of A is an equipotential surface. The potential of surface (1) is $\frac{Q}{r_1}$, that of (2) is $\frac{Q}{r_2}$ and that of the sphere A is $\frac{Q}{R}$.

The difference of potential between surfaces (1) and (2) is $\frac{Q}{r_1} - \frac{Q}{r_2}$ and is the work that must be done in taking a unit charge from any point on (2) to any point on (1). It makes no difference what path the charge follows, because its path can always be resolved into two displacements, one along the equipotential surface and the other normal to it; no work is done in moving

along the equipotential surface, since there is no opposing force and therefore all the work is done in the displacement along the normal.

The electrostatic force on an equipotential surface is normal to the surface, therefore lines of induction pass normally through equipotential surfaces.

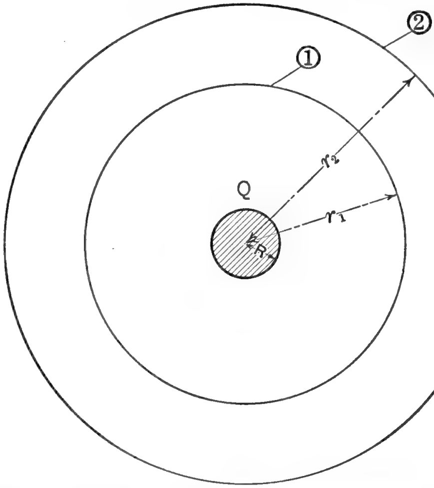


FIG. 9. Equipotential surfaces.

Electric conductors not carrying current are equipotential surfaces, since if differences of potential did exist electricity would flow from the points of high potential until the potential became uniform throughout the conductor.

17. Potential Gradient. The potential gradient at a point is the space rate of change of potential at the point measured in the direction of the electrostatic force.

The difference of potential between two points is

$$e = \int_0^D \mathfrak{F}\, dr, \qquad (12)$$

where $\mathfrak{F}$ is the electrostatic force at any point and D is the distance between the points. The potential gradient at any point is

$$g = \frac{de}{dr} = \mathfrak{F} \qquad (13)$$

and is equal to the electrostatic force at the point.

18. Potential and Potential Gradient in Special Cases. (a) Determine the potential and potential gradient in the field surrounding an isolated sphere of radius R charged with Q units of electricity. Fig. 10.

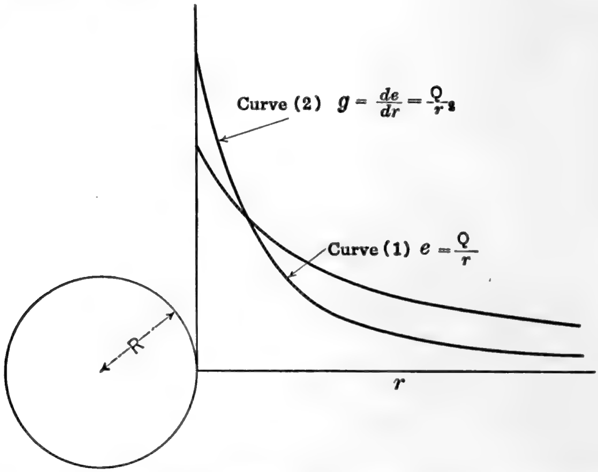


FIG. 10. Potential and potential gradient near a charged sphere.

The potential at a point P, distant r cm. from the centre of the sphere is

$$e = \frac{Q}{r}.$$

The potential gradient at P is

$$g = \frac{de}{dr} = -\frac{Q}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and is equal to the electrostatic force at the point.

Curve 1, Fig. 10, shows the variation of potential with distance measured from the centre of the sphere. The equation of this curve is

$$e = \frac{Q}{r}.$$

Across the sphere the potential is constant and is equal to $\frac{Q}{R}$, but near the surface it falls off very rapidly.

The potential gradient at any point is represented by the slope of the tangent to the potential curve at that point; its values are

plotted in curve 2. The equation of this curve is

$$g = \frac{de}{dr} = -\frac{Q}{r^2}.$$

(b) Determine the potential and the potential gradient at any point between two spheres A and B (Fig. 11) of radius R cm. if the distance between their centres is D cm. and they have charges of Q and $-Q$ respectively. This last condition means that all the lines of dielectric flux which leave the sphere A fall on the sphere B.

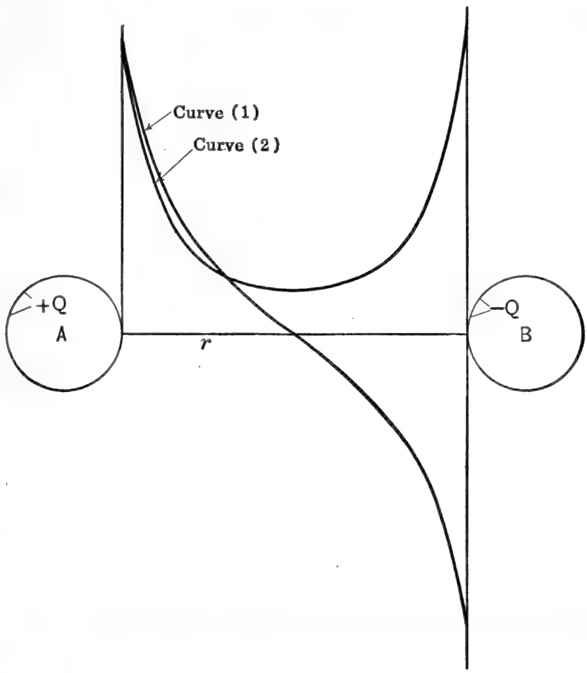


FIG. 11. Potential and potential gradient between two spheres.

At a point P on the line joining the centres of the two spheres and at a distance of r cm. from the centre of A the potential due to the charge Q on A is $\frac{Q}{r}$, and that due to the charge $-Q$ on B is $\frac{-Q}{D-r}$; the actual potential at P is, therefore,

$$e = \frac{Q}{r} - \frac{Q}{D-r}. \quad \dots \dots \dots \quad (15)$$

Midway between the spheres the potential is

$$\frac{Q}{D/2} - \frac{Q}{D - D/2} = O.$$

At the surface of A it is

$$\frac{Q}{R} - \frac{Q}{D - R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

The potentials at all points between A and B are plotted in curve 1 (Fig. 11).

The potential gradient at P is

$$g = \frac{de}{dr} = -\frac{Q}{r^2} - \frac{Q}{(D - r)^2} = -\left(\frac{Q}{r^2} + \frac{Q}{(D - r)^2}\right). \quad . \qquad (17)$$

Midway between the spheres the potential gradient is

$$-\left(\frac{Q}{(D/2)^2} + \frac{Q}{(D/2)^2}\right) = -\frac{8\,Q}{D^2}, \quad . \quad . \quad . \quad . \qquad (18)$$

which is its minimum value and at the surface of either sphere it is

$$\frac{Q}{R^2} + \frac{Q}{(D - R)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

its maximum value.

The potential gradient is plotted in curve 2, Fig. 11.

(c) Determine the potential and potential gradient at every point between two parallel cylindrical wires A and B, Fig. 12, of radius R, suspended in air with a distance of D cm. between centers and charged respectively with $+q$ and $-q$ units of electricity per centimeter length.

The electrostatic force at any point P distant r cm. from A and $D - r$ cm. from B is the resultant of the forces exerted by the charges on A and B considered as acting independently.

From each centimeter length of A, $4\,\pi q$ lines of dielectric flux extend out normally and produce at the point P a flux density

$$\mathfrak{D}_A = \frac{4\,\pi q}{2\,\pi r} = \frac{2\,q}{r} \text{ lines per square centimeter.} \quad . \quad . \qquad (20)$$

The electrostatic force at the point P is

$$\mathfrak{F}_A = \mathfrak{D}_A = \frac{2\,q}{r} \text{ dynes,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

and acts from A to B.

The charge $-q$ on B produces at P an electrostatic force

$$\mathfrak{F}_B = \frac{2q}{D-r} \text{ dynes,}$$

in the same direction as $\mathfrak{F}_A$.

The resultant force at P is

$$\mathfrak{F} = \mathfrak{F}_A + \mathfrak{F}_B = \frac{2q}{r} + \frac{2q}{D-r} \text{ dynes.} \quad . \quad . \quad . \quad (22)$$

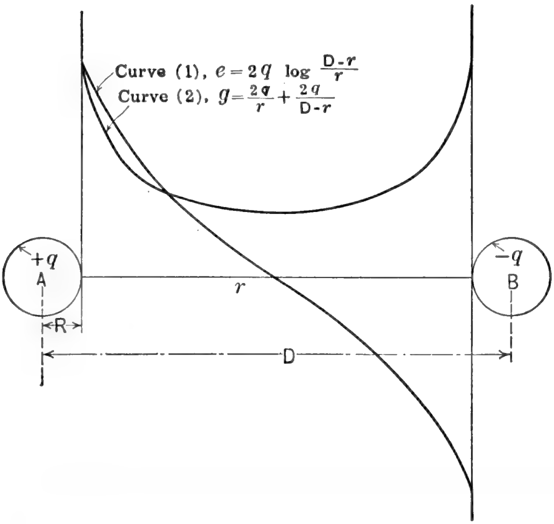


FIG. 12. Potential and potential gradient between two parallel wires.

The potential at the point midway between the two wires is zero and therefore the potential at the point P is

$$e = \int_r^{D/2} \mathfrak{F}\, dr = \int_r^{D/2} \left(\frac{2q}{r} + \frac{2q}{D-r} \right) dr$$

$$= 2q \Big[\log r - \log (D-r) \Big]_r^{D/2}$$

$$= 2q \log \frac{D-r}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This equation is plotted in curve 1, Fig. 12.

The potential at wire A is

$$E_A = 2q \log \frac{D-R}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and that at B is

$$E_B = -2q \log \frac{D-R}{R}, \quad \ldots \ldots \quad (25)$$

The difference of potential between the two wires is

$$E = E_A - E_B = 4q \log \frac{D-R}{R}. \quad \ldots \ldots \quad (26)$$

E can also be obtained as

$$E = \int_R^{D-R} \mathfrak{F}\, dr = 4q \log \frac{D-R}{R}.$$

The potential gradient between A and B, which is equal to the intensity of the electrostatic field, is given by equation

$$g = \mathfrak{F} = \frac{2q}{r} + \frac{2q}{D-r}, \quad \ldots \ldots \quad (27)$$

plotted as curve 2, Fig. 12.

At the surface of the wires its value is maximum.

$$g_{\text{max.}} = \frac{2q}{R} + \frac{2q}{D-R}. \quad \ldots \ldots \quad (28)$$

Midway between the wires its value is minimum.

$$g_{\text{min.}} = \frac{8q}{D}. \quad \ldots \ldots \quad (29)$$

If the two wires are suspended in a medium of dielectric constant K, the distribution of flux is not changed but the intensity of the electrostatic field and therefore the potential gradient and potential at each point are reduced in the ratio $1/K$, and the difference of potential between the wires becomes

$$E = \frac{4q}{K} \log \frac{D-R}{R}. \quad \ldots \ldots \quad (30)$$

19. Capacity. Conductors of different sizes and shapes have different capacities for storing electricity. If two conductors of different capacities are charged with equal quantities of electricity, they will be raised to different potentials, the one of small capacity will be raised to a high potential and the one of large capacity to a low potential.

The potential to which a body is raised varies directly as the quantity of electricity stored in it and inversely as its capacity and may be expressed by the formula

$$E = \frac{Q}{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where Q is the quantity of electricity, or charge,
C is the capacity of the body,
E is the potential.

The capacity therefore is

$$C = \frac{Q}{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and is equal to the charge divided by the potential, or it is equal to the charge per unit potential.

A body has unit capacity (electrostatic) when one unit of electricity is required to raise its potential by unity.

A sphere with a radius of one centimeter has unit capacity, because if a charge of one unit be given to it it will act as though it were concentrated at the centre and will produce at the surface unit potential. The capacity of a sphere of radius R cm. is R electrostatic units.

The practical unit of capacity is the farad. A conductor has a capacity of one farad when one coulomb of electricity is required to raise its potential by one volt.

$$1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} = \frac{3 \times 10^9}{\frac{1}{300}} = 9 \times 10^{11} \text{ electrostatic units. } (33)$$

The foregoing explanation of capacity is apt to be misleading. A conductor has capacity only with respect to surrounding objects, since the electrostatic energy is not stored on or in the conductor itself but in the field between the conductor and surrounding conductors.

In Fig. 13 A is a conductor placed near a large conducting plane B. Assume a potential difference E to exist between A and B, then E measures the work in ergs that must be done in carrying a unit charge from B to A against the electrostatic forces in the field. These forces produce a dielectric flux passing from A to B.

The total dielectric flux ψ is proportional to the difference of potential E and to the permeance of the path $\mathfrak{P}$.

The permeance $\mathfrak{P}$ is directly proportional to the cross-sectional area of the path and to its dielectric constant and is inversely proportional to the length of the path. It is usually difficult to calculate the permeance directly, since the section of the path varies throughout its length. In such cases the dielectric flux ψ or the electric charge $Q = \frac{\psi}{4\pi}$ is assumed and the potential difference E calculated as in the preceding examples.

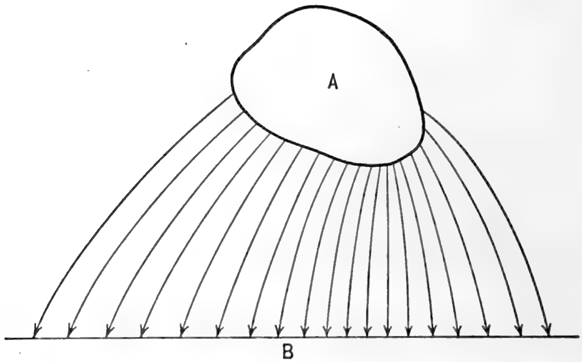


Fig. 13. Permeance of the path between a conductor and a plane.

The permeance is then

$$\mathfrak{P} = \frac{\psi}{E} = \frac{4\pi Q}{E} \quad \ldots\ldots \quad (34)$$

and the capacity is

$$C = \frac{Q}{E} = \frac{\psi}{4\pi E} = \frac{\mathfrak{P}}{4\pi}; \quad \ldots\ldots \quad (35)$$

thus the capacity is equal to the dielectric permeance of the path divided by 4π.

20. Condenser. — An electrical condenser is an arrangement of conductors which is capable of storing a large quantity of electricity, or which has a large capacity. It is one in which a large flux is produced when a given potential difference is applied to its terminals.

21. Capacity of Condensers. The capacity of a condenser is measured by the amount of charge necessary to raise its potential by unity, and is therefore equal to the positive charge on it at any time divided by the potential difference between its terminals;

this relation is expressed by the formula

$$C = \frac{Q}{E}.$$

(a) Determine the capacity of a condenser formed of two parallel plates each having an area of A square centimeters separated by t cm. of material of dielectric constant K. Fig. 14.

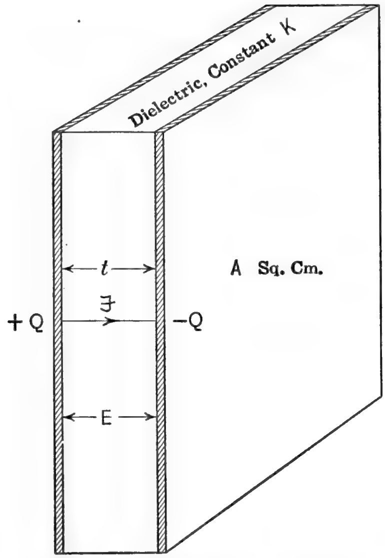


FIG. 14. Parallel plate condenser.

If a difference of potential E is applied between the terminals a dielectric flux ψ is produced passing from the plate of high potential to the plate of low potential; a positive charge $Q = \frac{\psi}{4\pi}$ therefore appears on one plate and a negative charge $-Q = -\frac{\psi}{4\pi}$ on the other; the density of the charge on the plates is $q = \frac{Q}{A}$ units per square centimeter, and the dielectric flux density between the plates is $\mathfrak{D} = 4\pi q = \frac{4\pi Q}{A} = \frac{\psi}{A}$ lines per square centimeter. The thickness of the dielectric is assumed to be so small that the lines of flux travel directly across from one plate to the other.

The intensity of the electrostatic force is

$$\mathfrak{F} = \frac{\mathfrak{D}}{K} = \frac{4\,\pi Q}{AK}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The difference of potential between the plates is the work done in carrying a unit charge from one plate to the other against the force $\mathfrak{F}$; it is therefore

$$E = \mathfrak{F}t = \frac{4\,\pi Qt}{AK}$$

since $\mathfrak{F}$ is constant; and the capacity of the condenser is

$$C = \frac{Q}{E}$$

$$= \frac{Q}{\dfrac{4\,\pi Qt}{AK}} = \frac{AK}{4\,\pi t} \text{ electrostatic units.} \quad . \quad . \quad . \quad . \quad (37)$$

The capacity therefore varies directly with the area of the plates and with the dielectric constant of the material separating them, and inversely as the distance between them.

When the plates are separated by air the capacity is

$$C = \frac{A}{4\,\pi t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

In order to increase the capacity of such a condenser a large

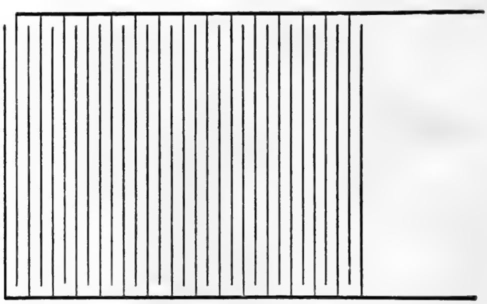


FIG. 15. Electric condenser.

number of plates are used joined in multiple and separated by very thin sheets of dielectric as shown in Fig. 15.

(b) Determine the capacity of a condenser formed of two concentric cylinders, Fig. 16, of radii a and b.

If a charge of q units per centimeter length is given to the inner cylinder, lines of dielectric flux pass out radially and produce, at a distance r cm. from the axis of the cylinder, a flux density

$$\mathfrak{D} = \frac{4\pi q}{2\pi r} = \frac{2q}{r} \text{ lines per square centimeter.}$$

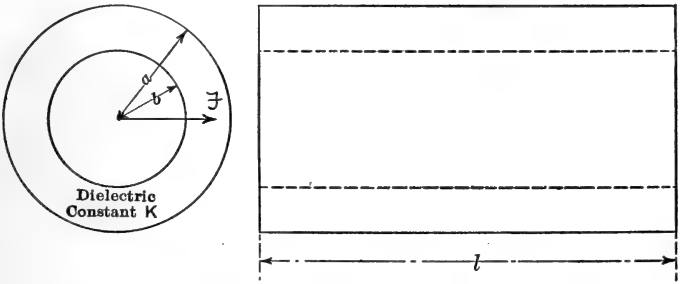


FIG. 16. Capacity of concentric cylinders.

If the dielectric constant of the material is K, the electrostatic force at the point is

$$\mathfrak{F} = \frac{\mathfrak{D}}{K} = \frac{2q}{Kr} \text{ dynes,}$$

and the difference of potential between the two cylinders is

$$E = \int_b^a \frac{2q}{K}\frac{dr}{r} = \frac{2q}{K}\log\frac{a}{b} \text{ ergs.} \quad . \quad . \quad . \quad . \quad (39)$$

The capacity per centimeter length of the condenser is

$$C = \frac{q}{E} = \frac{q}{\frac{2q}{K}\log\frac{a}{b}} = \frac{K}{2\log\frac{a}{b}} \text{ electrostatic units.} \quad . \quad (40)$$

This is the case of a single conductor cable with a lead sheath, and the capacity is usually expressed in farads per mile; it is

$$C = \frac{K}{2\log\frac{a}{b}} \times \frac{2.54 \times 12 \times 5280}{9 \times 10^{11}}$$

$$= \frac{K}{2 \times 2.303 \log_{10}\frac{a}{b}} \times \frac{2.54 \times 12 \times 5280}{9 \times 10^{11}}$$

$$= 3.82 \times \frac{K}{\log_{10}\frac{a}{b}} \times 10^{-6} \text{ farads per mile.} \quad . \quad . \quad (41)$$

(c) Determine the capacity of two parallel wires A and B in Fig. 17.

When a difference of potential E is applied between them a dielectric flux passes across from A to B and a positive charge $Q = \frac{\psi}{4\pi}$ appears on A and an equal negative charge $-Q = -\frac{\psi}{4\pi}$ on B.

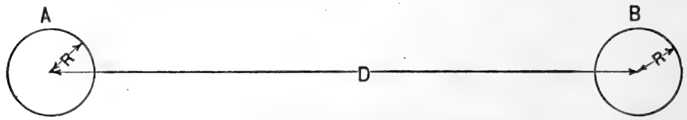


FIG. 17. Capacity of parallel wires.

If q is the charge per centimeter length on A, the potential difference between A and B is

$$E = 4\,q \log \frac{D-R}{R}, \text{ Eq. 26,}$$

where R is the radius of the wires and D is the distance between centres.

The capacity of the two wires per centimeter length is

$$C = \frac{q}{E} = \frac{q}{4\,q \log \frac{D-R}{R}} = \frac{1}{4 \log \frac{D-R}{R}} \text{ electrostatic units.} \quad (42)$$

The capacity per mile of a line consisting of two wires is

$$C = \frac{2.54 \times 12 \times 5280}{4 \times 2.303 \log_{10} \frac{D-R}{R} \times 9 \times 10^{11}}$$

$$= \frac{19.4 \times 10^{-9}}{\log_{10} \frac{D-R}{R}} \text{ farads.} \quad (43)$$

It is sometimes useful to separate the capacity of the line into the capacities of the two wires forming it. The potential of the point midway between the wires is zero and the actual potential of wire A is

$$E_A = 2\,q \log \frac{D-R}{R}, \text{ Eq. 24,}$$

and therefore the capacity of A per centimeter length between the wire and the neutral or point of zero potential is

$$C_A = \frac{q}{E_A} = \frac{q}{2\,q\log\dfrac{D-R}{R}} = \frac{1}{2\log\dfrac{D-R}{R}}\ \text{electrostatic units,} \quad (44)$$

and the capacity of each wire in farads per mile is

$$C = \frac{38.8 \times 10^{-9}}{\log_{10}\dfrac{D-R}{R}}. \quad \ldots \ldots \quad (45)$$

(d) Determine the capacity of a single wire of radius R cm., suspended at a height H cm. above the earth. If the wire is raised to a potential E above the earth potential, lines of dielectric flux will pass from the wire to earth as shown in Fig. 18.

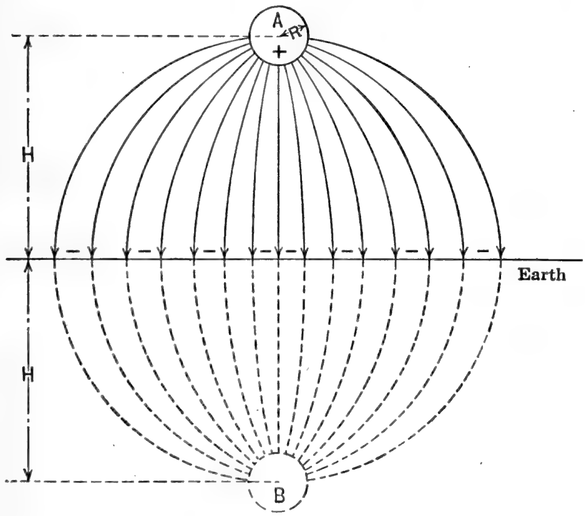


FIG. 18. Capacity of a wire to earth

If ψ is the flux from each centimeter length of A, then $q = \dfrac{\psi}{4\pi}$ is the charge on each centimeter of A, and a corresponding negative charge $-q$ appears on the earth, but it is not evenly distributed being most concentrated directly beneath the wire.

The flux passing from the wire A to the earth would be unchanged if the charge on the earth were collected on a second

wire B placed at a distance H cm. below the earth. Its potential would be $-E$.

The difference of potential between A and B is $2E$ and from equation 26

$$2E = 4q \log \frac{2H - R}{R},$$

thus the capacity of the single wire in relation to the earth per centimeter length is

$$C = \frac{q}{E} = \frac{q}{2q \log \frac{2H-R}{R}} = \frac{1}{2 \log \frac{2H-R}{R}} \text{ e.s. units}, \quad (46)$$

and the capacity in farads per mile is

$$C = \frac{38.8 \times 10^{-9}}{\log_{10} \frac{2H-R}{R}}. \quad (47)$$

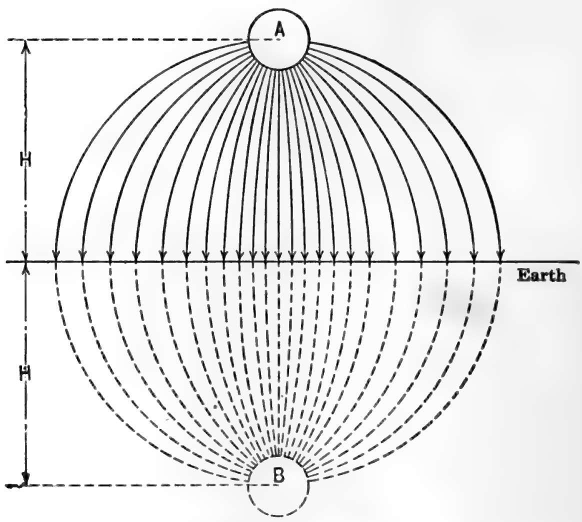


Fig. 19. Capacity of sphere to earth.

(e) Capacity of a sphere of radius R cm. at height H cm. above the earth. In Fig. 19 A is a sphere of radius R cm. at a height H cm. above the earth. If A is raised to a potential E above earth potential, a dielectric flux ψ is produced passing from the

sphere to the earth as shown and a positive charge $Q = \frac{\psi}{4\pi}$ appears on A and an equal negative charge on the earth.

Without changing the distribution of flux in any way the negative charge on the earth may be assumed to be collected on a sphere B similar to A but placed at a distance H cm. below the surface. The distribution of potential between two such spheres was worked out in Art. 18. The potential midway between them is zero, the potential at the surface of A is

$$E = \frac{Q}{R} - \frac{Q}{2H - R} \text{ by equation 16,}$$

and therefore the capacity of the sphere is

$$C = \frac{Q}{E} = \frac{Q}{\frac{Q}{R} - \frac{Q}{2H - R}} = \frac{1}{\frac{1}{R} - \frac{1}{2H - R}} \text{ electrostatic units.} \quad (48)$$

When H is very large compared to R the capacity is

$$C = \frac{1}{1/R} = R \text{ as in Art. 19.}$$

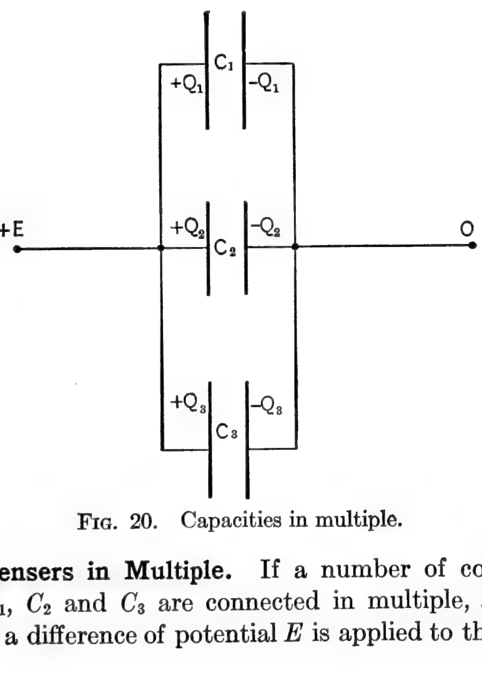


FIG. 20. Capacities in multiple.

22. Condensers in Multiple. If a number of condensers of capacities C_1, C_2 and C_3 are connected in multiple, as shown in Fig. 20, and a difference of potential E is applied to the terminals,

each condenser receives a charge proportional to its capacity,

$$Q_1 = C_1E,$$
$$Q_2 = C_2E,$$
$$Q_3 = C_3E,$$

and the total charge on the system is

$$Q = Q_1 + Q_2 + Q_3 = E\,(C_1 + C_2 + C_3).$$

The capacity of the system is

$$C = \frac{Q}{E} = C_1 + C_2 + C_3, \quad . \quad . \quad . \quad . \quad . \quad (49)$$

and therefore the capacity of a number of condensers connected in multiple is equal to the sum of their separate capacities.

23. Condensers in Series. When a number of condensers of capacities C_1, C_2 and C_3 are connected in series, as in Fig. 21, and

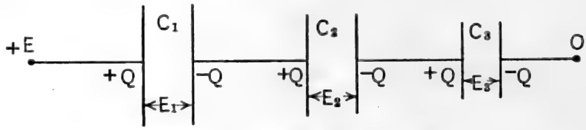


FIG. 21. Capacities in series.

a difference of potential E is applied to the terminals of the system, a charge Q appears on each condenser and the potential E divides up among the condensers in inverse proportion to their capacities.

The drop of potential across condenser (1) is

$$E_1 = \frac{Q}{C_1},$$

that across (2) is

$$E_2 = \frac{Q}{C_2},$$

and that across (3) is

$$E_3 = \frac{Q}{C_3};$$

but

$$\begin{aligned} E &= E_1 + E_2 + E_3 \\ &= \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} \\ &= Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right), \end{aligned}$$

and, therefore, the capacity of the system consisting of three condensers in series is

$$C = \frac{Q}{E} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}. \quad \ldots \quad (50)$$

When two equal condensers of capacity C are connected in series, their joint capacity is

$$C_1 = \frac{1}{\frac{1}{C} + \frac{1}{C}} = \frac{C}{2}, \quad \ldots \quad (51)$$

and is equal to one half of the capacity of either condenser alone.

24. Energy Stored in a Condenser. When a condenser is being charged, work is done in raising the charge through the difference of potential between the terminals, and this amount of energy is stored in the electrostatic field of the condenser.

In Fig. 22 PN is a condenser of capacity C formed of two parallel plates separated by t cm. of a dielectric of constant K. When a potential difference e is produced between the plates by the generator G, electricity flows from N to P until the charge on P is

$$q = Ce;$$

if the potential e is increased by de the charge q is increased by $dq = C\,de$ and the work done in raising the charge dq through the difference of potential e is

$$d\omega = e\,dq = Ce\,de.$$

The total work done in charging the condenser with a quantity of electricity Q, or to a difference of potential E, is

$$\begin{aligned} W &= \int d\omega = \int_0^E Ce\,de \\ &= C\left[\frac{e^2}{2}\right]_0^E \\ &= C\frac{E^2}{2} \text{ ergs.} \quad \ldots \quad (52) \end{aligned}$$

Thus the work done in charging a condenser, or the energy stored in the electrostatic field of the condenser, is equal to one half of the capacity multiplied by the square of the difference of potential between the terminals.

Equation 52 may be expressed in two other forms by substituting for E its value $\frac{Q}{C}$;

$$W = \frac{1}{2} QE, \quad \ldots \ldots \ldots \quad (53)$$

or

$$W = \frac{1}{2} \frac{Q^2}{C}. \quad \ldots \ldots \ldots \quad (54)$$

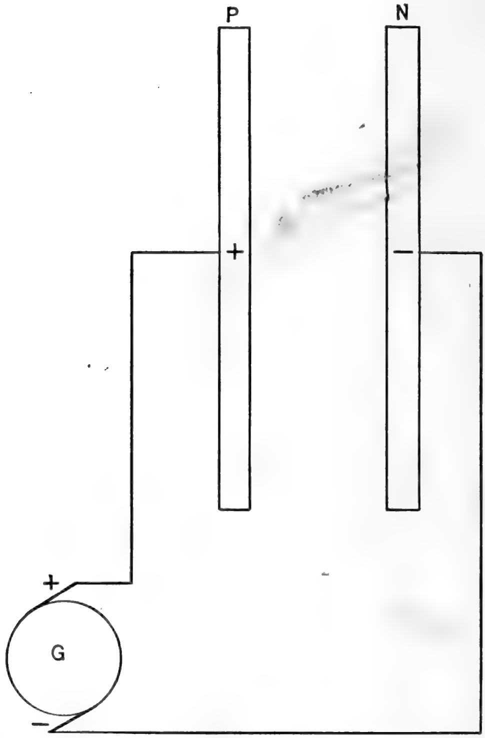


Fig. 22. Energy stored in a condenser.

If the area of the plate P is A square centimeters, then the dielectric flux density between the plates is

$$\mathfrak{D} = \frac{4\pi Q}{A} \text{ lines per square centimeter,}$$

and the electrostatic force is

$$\mathfrak{F} = \frac{\mathfrak{D}}{K} \text{ dynes.}$$

The potential difference between the plates is

$$E = \mathcal{F}t = \frac{\mathcal{D}t}{K},$$

and the capacity of the condenser by equation 37 is

$$C = \frac{AK}{4\pi t}.$$

Substituting these values for E and C in equation 52 gives a fourth expression for the energy stored in the field, namely

$$W = \frac{1}{2} \times \frac{AK}{4\pi t} \times \left(\frac{Dt}{K}\right)^2$$

$$= At \times \frac{\mathcal{D}^2}{8\pi K} \text{ ergs.} \qquad (55)$$

Since the volume of the field is At cubic centimeters and the flux density is uniform, the energy stored per cubic centimeter of the field is

$$\omega = \frac{\mathcal{D}^2}{8\pi K} \text{ ergs} \qquad (56)$$

or

$$\omega = \frac{\mathcal{F}^2 K}{8\pi} \text{ ergs.} \qquad (57)$$

Thus, the energy stored per cubic centimeter in an electrostatic field is equal to the square of the dielectric flux density multiplied by $\frac{1}{8\pi K}$, or is equal to the square of the intensity of the electrostatic force multiplied by $\frac{K}{8\pi}$.

From equation 52 a very useful definition of capacity may be obtained,

$$C = \frac{2W}{E^2}, \qquad (58)$$

or the capacity of a condenser is equal to twice the energy stored in its field divided by the square of the difference of potential across its terminals, or the capacity is equal to twice the energy stored when the difference of potential is unity.

25. Stresses in an Electrostatic Field. The energy stored in an electrostatic field is

$$W = C\frac{E^2}{2} \text{ ergs;}$$

and the energy stored per cubic centimeter is

$$\omega = \frac{\mathfrak{D}^2}{8\,\pi K}.$$

These two equations represent the potential energy of the field. Stresses exist throughout the field tending to reduce the potential energy to a minimum; first, there is a tension along the lines of induction tending to shorten them and to draw the bounding surfaces of the field together and so reduce the volume to zero; second, there is a pressure at right angles to the lines tending to spread them apart and so reduce the density in the field. Since the system is in equilibrium these two stresses are of equal magnitude.

To obtain an expression for the stress per square centimeter on the bounding surfaces, consider the parallel plate condenser in Fig. 23. The energy stored in the field is by equation 55

$$W = At \times \frac{\mathfrak{D}^2}{8\,\pi K}\text{ ergs.}$$

If a force of P dynes is applied to one of the plates and the distance between the plates is increased by amount dt, the work done is $P\,dt$ ergs. The charges on the plates are assumed to remain constant and therefore the flux density remains constant, but the volume of the field is increased by the amount $A\,dt$ cu. cm., and the energy stored in it is increased by $A\,dt \times \frac{\mathfrak{D}^2}{8\,\pi K}$, but the increase in the stored energy is equal to the work done by the force P and, therefore,

$$P\,dt = A\,dt \times \frac{\mathfrak{D}^2}{8\,\pi K}$$

and

$$P = A \times \frac{\mathfrak{D}^2}{8\,\pi K}\text{ dynes.} \qquad (59)$$

This is the pull exerted by the field on each plate of the condenser tending to draw them together.

The pull per square centimeter is

$$p = \frac{P}{A} = \frac{\mathfrak{D}^2}{8\,\pi K}\text{ dynes,} \qquad (60)$$

thus, the pull per square centimeter on any charged surface is equal to the square of the induction density at the point divided by $8\,\pi K$.

This is the value of the tension along the lines of force tending to shorten them, and also the value of the pressure at right angles to the lines tending to spread them apart.

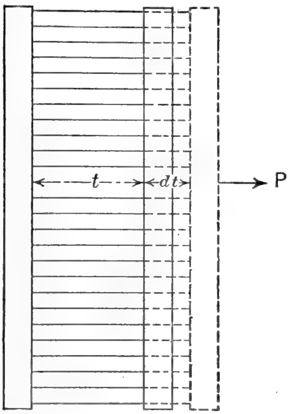


FIG. 23. Stresses in the electrostatic field.

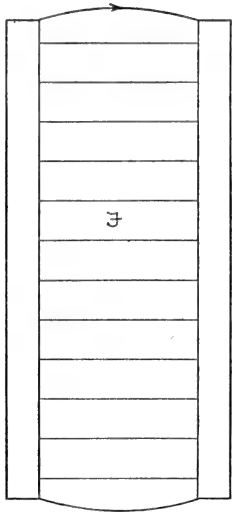


FIG. 24.

26. Effects of Introducing Dielectrics of Various Specific Inductive Capacities into a Uniform Field. (1) Fig. 24 shows a parallel plate condenser with air as dielectric.

t = distance between plates in centimeters.
A = area of each plate in square centimeters.
E = difference of potential between plates.
ψ = dielectric flux.

$$\mathfrak{D} = \frac{\psi}{A} = \text{dielectric flux density.}$$

$$\mathfrak{F} = \mathfrak{D} = \text{electrostatic force in the field.}$$

Since the force is constant throughout the field, therefore,

$$E = \mathfrak{F}t$$

and the electrostatic force or the stress in the air is

$$\mathfrak{F} = \frac{E}{t}.$$

(2) In Fig. 25 a sheet of glass of thickness $0.9\,t$ and dielectric constant $K = 6$ is introduced into the field as shown and the difference of potential is the same as before.

The dielectric flux density $\mathcal{D}$ is constant throughout the field; the electrostatic force in the air is

$$\mathcal{F}_A = \mathcal{D};$$

the drop of potential across the air portion of the field is

$$E_1 = 0.1\,t \times \mathcal{F}_A = 0.1\,t \times \mathcal{D};$$

the electrostatic force in the glass is

$$\mathcal{F}_G = \frac{\mathcal{D}}{K} = \frac{\mathcal{D}}{6};$$

the drop of potential across the glass is

$$E_2 = 0.9\,t \times \mathcal{F}_G = 0.9\,t \times \frac{\mathcal{D}}{6} = 0.15\,t \times \mathcal{D};$$

the difference of potential between the plates is

$$\begin{aligned} E &= E_1 + E_2 \\ &= 0.1\,t \times \mathcal{D} + 0.5\,t \times \mathcal{D} \\ &= 0.25\,t\mathcal{D}, \end{aligned}$$

and the dielectric flux density is

$$\mathcal{D} = \frac{E}{0.25\,t} = 4\frac{E}{t};$$

thus the stress in the air is

$$\mathcal{F}_A = \mathcal{D} = 4\frac{E}{t},$$

and is four times as great as it was in the first case.

(3) Fig. 26 shows the same pair of plates with three sheets of dielectric introduced between them, of thickness t_1, t_2 and t_3 and dielectric constants K_1, K_2 and K_3 respectively.

$\mathcal{D}$ = dielectric flux, which is constant throughout the field.

$\mathcal{F}_1 = \dfrac{\mathcal{D}}{K_1}$ = stress in layer (1),

$\mathcal{F}_2 = \dfrac{\mathcal{D}}{K_2}$ = stress in layer (2),

$\mathcal{F}_3 = \dfrac{\mathcal{D}}{K_3}$ = stress in layer (3),

$E_1 = \mathcal{F}_1 t_1 = \dfrac{\mathcal{D}}{K_1} t_1$ = drop of potential across (1),

$$E_2 = \mathfrak{F}_2 t_2 = \frac{\mathfrak{D}}{K_2} t_2 = \text{drop of potential across (2)},$$

$$E_3 = \mathfrak{F}_3 t_3 = \frac{\mathfrak{D}}{K_3} t_3 = \text{drop of potential across (3)};$$

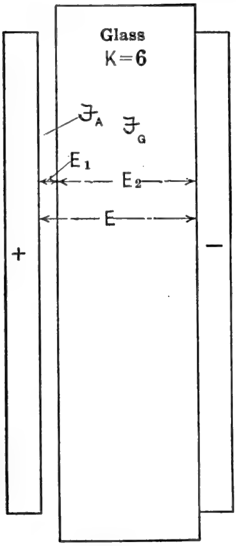


FIG. 25.

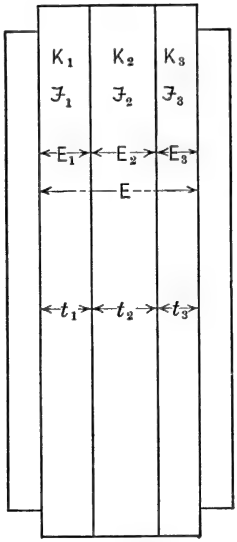


FIG. 26.

the difference of potential between the plates is

$$E = E_1 + E_2 + E_3$$
$$= \mathfrak{D}\left(\frac{t_1}{K_1} + \frac{t_2}{K_2} + \frac{t_3}{K_3}\right),$$

and the dielectric density is

$$\mathfrak{D} = \frac{E}{\frac{t_1}{K_1} + \frac{t_2}{K_2} + \frac{t_3}{K_3}},$$

and the total flux is

$$\psi = \mathfrak{D}A = \frac{E}{\frac{t_1}{K_1} + \frac{t_2}{K_2} + \frac{t_3}{K_3}} A = E\mathfrak{P},$$

where

$$\mathfrak{P} = \frac{\psi}{E} = \frac{A}{\frac{t_1}{K_1} + \frac{t_2}{K_2} + \frac{t_3}{K_3}}, \text{ Eq. 34,}$$

is the permeance of the path between the plates.

From the last two examples it is seen that a uniform field can be made non-uniform by introducing dielectrics of various specific inductive capacities, and that the stress in the various dielectrics varies inversely as their specific inductive capacities.

(4) In Fig. 27 a cylinder of dielectric constant K is introduced into the uniform field as shown.

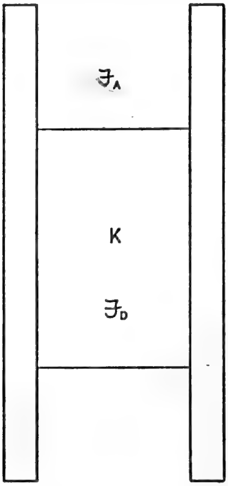


Fig. 27.

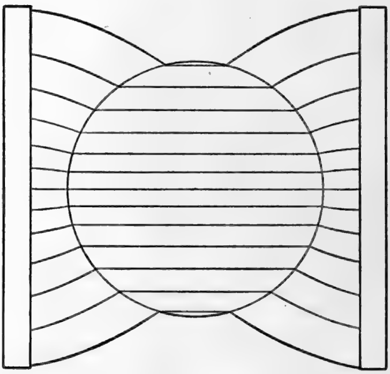


Fig. 28.

The stress in the air is

$$\mathfrak{F}_A = \frac{E}{t},$$

and is the same as before the cylinder was introduced.

The stress in the dielectric is also

$$\mathfrak{F}_D = \mathfrak{F}_A = \frac{E}{t},$$

but the induction density in the dielectric is

$$\mathfrak{D} = \mathfrak{F}_D K = \frac{E}{t} K.$$

The permeance of the path is increased and an increased flux is produced passing between the plates, but since the increased flux is all confined to the cylinder there is no increase in the stress in the air.

(5) If a sphere of dielectric constant K is placed in the field, as shown in Fig. 28, the permeance of the path between the plates is increased, but the increase of flux is not confined to the sphere. Where the sphere approaches nearest to the plates the density is very great, since a large number of lines take the short path through the air in order to follow the long path through the sphere. The stress in the air in these regions is very largely increased.

(6) In Fig. 29 A is a small piece of material with a high dielectric constant such as a drop of oil. It offers a local path of high

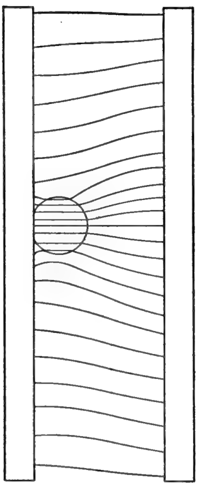


FIG. 29.

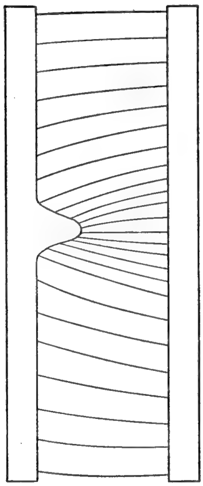


FIG. 30.

permeability, and a greater flux density is produced in it than in other parts of the field, and therefore the stress in the air at its surface will be greater than the average stress throughout the field.

(7) In Fig. 30 the drop of oil is replaced by a knob on the conductor forming one boundary of the field. Since the dielectric constant of the conducting knob is infinity the stress in the air at its surface will be greater than in the case of the oil.

(8) Fig. 31 shows the field at the edge of the condenser in case (1). In this region the dielectric flux density is not uniform, and just at the edges of the plates it is greater than in the main body

of the field. The electrostatic stress is also greater than the average value.

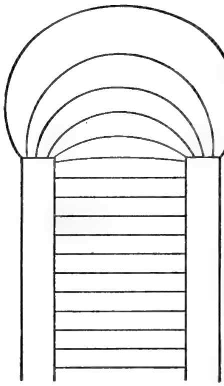


Fig. 31.

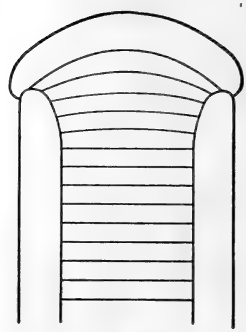


Fig. 32.

This condition may be corrected by rounding off the edges of the plates or electrodes as shown in Fig. 32.

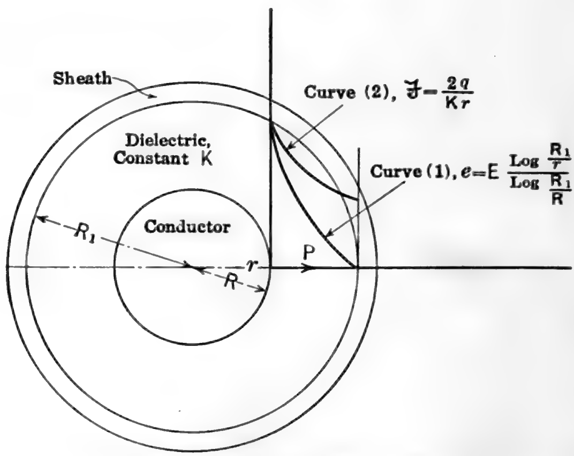


Fig. 33. Potential and potential gradient in a lead covered cable.

27. Graded Insulation for Cables. Fig. 33 shows a single conductor, lead covered cable insulated with material of dielectric constant K.

R cm. = radius of the conductor,

R_1 cm. = inside radius of the sheath.

If the conductor is raised to a potential E and the sheath is grounded, the potential at any point P distant r cm. from the centre of the conductor may be found as follows:

Assume that the charge per centimeter length of the conductor is q units, then the flux density at P is

$$= \frac{2q}{r} \text{ lines per square centimeter,}$$

and the electrostatic force or stress in the medium is

$$\mathfrak{F} = \frac{\mathfrak{D}}{K} = \frac{2q}{Kr} \text{ dynes;}$$

the difference of potential between the conductor and the sheath is

$$E = \int_R^{R_1} \frac{2q}{Kr}\,dr = \frac{2q}{K}\log\frac{R_1}{R},$$

and therefore the charge per centimeter is

$$q = \frac{KE}{2\log\frac{R_1}{R}}; \qquad (61)$$

the potential of P is

$$e = \int_r^{R_1} \frac{2q}{Kr}\,dr = \frac{2q}{K}\log\frac{R_1}{r},$$

and substituting for q its value from equation 61,

$$e = E\frac{\log\frac{R_1}{r}}{\log\frac{R_1}{R}}; \qquad (62)$$

this equation is plotted in curve 1, Fig. 33, and represents the potential at all points between the conductor and sheath. The potential gradient or electrostatic stress in the medium at the point P is

$$\mathfrak{F} = \frac{2q}{Kr} = \frac{E}{r\log\frac{R_1}{R}}, \qquad (63)$$

and varies inversely as the distance from the centre of the conductor. Its values are plotted in curve 2.

The stresses in the dielectric are not uniform, but are greatest near the conductor and least in the outer layers near the sheath. The outer portion of the dielectric is therefore not used to the best advantage.

Fig. 34 shows the same cable insulated with three layers of material with dielectric constants K_1, K_2 and K_3.

The dielectric flux density at distance r cm. from the centre of the core is

$$\mathcal{D} = \frac{2q}{r} \text{ lines per square centimeter,}$$

and the stress is

$$\mathcal{F} = \frac{2q}{Kr} \text{ dynes,}$$

where k has different values in the three dielectrics.

In order to make the stresses in the various parts of the insulation equal it would be necessary to place next to the conductor a

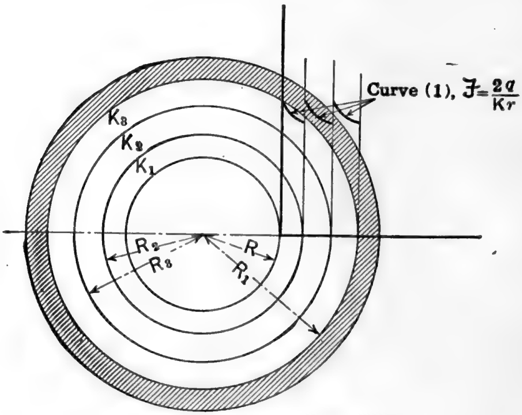


Fig. 34. Graded insulation.

material of high dielectric constant K_1 and to gradually decrease this constant in succeeding layers in inverse proportion to the distance from the centre of the conductor. This is a very expensive process and is not necessary since good results can be obtained by using three or four layers of dielectric.

If R_3 and R_2 are the inside radii of the two outer layers, the stress in the outer layer is

$$\mathcal{F}_3 = \frac{2q}{K_3 r},$$

and the drop of potential across it is

$$E_3 = \int_{R_3}^{R_1} \frac{2q}{K_3 r} dr = \frac{2q}{K_3} \log \frac{R_1}{R_3}; \quad . \quad . \quad . \quad . \quad (64)$$

the stress in the second layer is

$$\mathfrak{F}_2 = \frac{2\,q}{K_2 r},$$

and the drop of potential across it is

$$E_2 = \int_{R_2}^{R_3} \frac{2\,q}{K_2 r} = \frac{2\,q}{K_2} \log \frac{R_3}{R_2};$$

the stress in the inner layer is

$$\mathfrak{F}_1 = \frac{2\,q}{K_1 r},$$

and the drop of potential across it is

$$E_1 = \int_{R}^{R_2} \frac{2\,q}{K_1 r}\,dr = \frac{2\,q}{K_1} \log \frac{R_2}{R}.$$

The difference of potential between the conductor and the sheath is

$$\begin{aligned} E &= E_1 + E_2 + E_3 \\ &= \frac{2\,q}{K_1} \log \frac{R_2}{R} + \frac{2\,q}{K_2} \log \frac{R_3}{R_2} + \frac{2\,q}{K_3} \log \frac{R_1}{R_3}. \quad \ldots \quad (65) \end{aligned}$$

If K_1, K_2 and K_3 are chosen inversely proportional to the inside radii of the three layers, then the stress at these points will be equal and will be the maximum stress in the field and

$$\mathfrak{F}_{\max} = \frac{2\,q}{K_1 R} = \frac{2\,q}{K_2 R_2} = \frac{2\,q}{K_3 R_3}. \quad \ldots \ldots \quad (66)$$

Curve 1, Fig. 34, shows the variation of potential gradient or electrostatic stress from the conductor to the sheath.

The stress in the dielectric is much more uniform than in the case of the same cable insulated with a single dielectric.

The thicknesses of the three layers have been assumed to be equal.

From these examples it is seen that a field in which the stresses are not uniform can be made more uniform by grading the insulation, that is, using a material of high dielectric constant in the part of the field where the stress tends to be a maximum and gradually decreasing the dielectric constant in the parts where the stress tends to be low.

From equation 65 the charge per centimeter length of the con-

ductor is found to be

$$q = \frac{E}{2}\left(\frac{1}{K_1 \log \frac{R_2}{R}} + \frac{1}{K_2 \log \frac{R_3}{R_2}} + \frac{1}{K_3 \log \frac{R_1}{R_3}}\right), \quad . \quad . \quad (67)$$

and the capacity of the cable per centimeter length is

$$C = \frac{q}{E} = \frac{1}{2}\left(\frac{1}{K_1 \log \frac{R_2}{R}} + \frac{1}{K_2 \log \frac{R_3}{R_2}} + \frac{1}{K_3 \log \frac{R_1}{R_3}}\right). \quad . \quad (68)$$

28. Dielectric Strength. As the difference of potential between two insulated conductors is increased the intensity of the electrostatic field between them increases and the energy stored in it increases. At a certain point the stresses in the dielectric become so great that a rupture occurs and a discharge takes place between the electrodes and the energy disappears from the field.

If the dielectric is a gas or a liquid any effect of the discharge is remedied by circulation of the dielectric. When a heavy discharge takes place through oil or other similar material it may become partially carbonized, and the carbon particles tend to line up in the intense field between the electrodes and form a conducting bridge.

In the case of solid dielectrics a rupture occurs which destroys the insulating properties of the dielectric.

The difference of potential or electromotive force between terminals at which the breakdown occurs depends on the dielectric material, on the distance between electrodes and on the distribution of the electrostatic flux in the dielectric, that is, on the shape of the electrodes. Breakdown occurs due to high electrostatic stresses in the medium or due to a steep potential gradient.

The potential gradient is usually expressed in volts per inch or volts per centimeter, but since even one centimeter of dielectric requires a very high voltage to puncture it the dielectric strength may be defined as the difference of potential in volts required to cause a discharge through one millimeter thickness of the material but the shapes of the electrodes should always be specified.

In the following table the approximate dielectric strengths of various insulating materials are given:

Dielectric.	Dielectric strength in volts per mm.
Mica	58,000
Micanite	35,000
Paraffined paper	30,000
Porcelain	13,000
Dry wood fibre	13,000
Oiled linen	12,500
Vulcanized rubber	10,000
Transformer oil	9,000
Air	3,500

29. Corona. When the electrostatic stress or potential gradient at some part of the surface of a conductor in air exceeds about 3500 volts per millimeter or 100,000 volts per inch a brush discharge takes place and the air becomes conducting and luminous. The discharge does not necessarily extend across from the positive to the negative electrode but only exists in the region where the dielectric strength of the air has been exceeded. Take the case of two parallel wires suspended in air; the discharge or corona first appears at any rough points on the wires, and finally forms a luminous envelope about them, which increases in diameter as the voltage is raised. Corona causes a loss of energy which increases very rapidly with increase of voltage above the critical point where the discharge begins.

CHAPTER II

MAGNETISM AND ELECTROMAGNETICS

30. Magnetization. When bodies are magnetized magnetic forces act at every point throughout their volume and lines of magnetic induction pass through them. There are two kinds of magnetic poles just as there are two kinds of electric charges; as a positive electric charge appears where a dielectric flux leaves a surface, so a positive magnetic pole appears where a magnetic flux leaves a surface. The positive magnetic pole is called a north pole. Similarly a negative magnetic pole or south pole appears where a magnetic flux enters a surface.

Thus a body which is magnetized has a north pole at one part of its surface and an equal south pole at another part unless the magnetic path forms a closed circuit as in Fig. 36.

Fig. 35 represents a horseshoe magnet. The lines of magnetic induction pass through it in the direction shown, leaving the surface at N and entering it again at S. Thus N is a positive magnetic pole or a north pole and S is a negative magnetic pole or a south pole.

Fig. 36 represents the same magnet with its armature on. The lines of magnetic induction pass in the same direction as before, but the circuit is closed and the poles do not appear until a gap is made by removing the armature.

31. Laws of Magnetism. *First Law.* Like magnetic poles repel one another; unlike magnetic poles attract one another.

Second Law. The force exerted between two magnetic poles is proportional to the product of their strengths and is inversely proportional to the square of the distance between them. This law can be expressed by the formula

$$f = \frac{mm_1}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

where m and m_1 are the pole strengths, r is the distance between them in centimeters, and f is the force exerted between them in dynes. If m and m_1 are like poles the force is a repulsion and f is positive.

The unit of pole strength is defined as follows: A magnetic pole has unit strength if, when placed at a distance of one centimeter from a similar pole, it repels it with a force of one dyne.

The force exerted on a unit pole at a distance of r cm. from a pole of strength m is

$$f = \frac{m}{r^2} \text{ dynes.} \qquad (70)$$

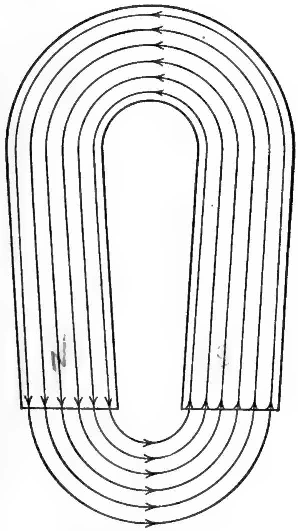


FIG. 35. Magnet.

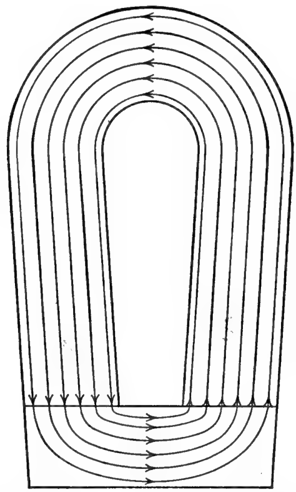


FIG. 36. Magnet with armature on.

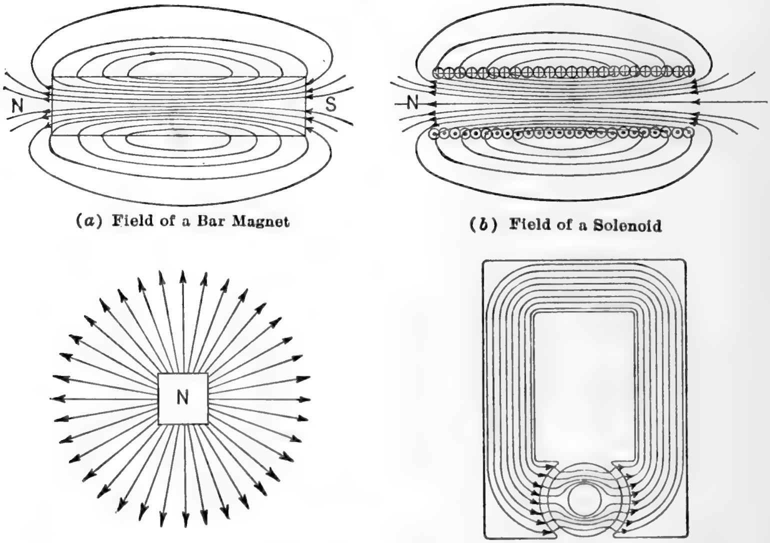
32. Magnetic Field. The space surrounding a magnetic pole or a current of electricity in which magnetic forces act is called a magnetic field. The direction of the force at any point in the field is the direction in which a unit north pole placed at the point would tend to move and its intensity is the force in dynes exerted on the unit pole.

The magnetic field is represented by lines of magnetic induction or magnetic flux drawn in the direction of the force.

Unit magnetic force or unit magnetizing force produces one line of magnetic flux per square centimeter in air and μ lines per square centimeter in a magnetic material of permeability μ.

The magnetic force at a point is expressed in dynes and is represented by $\mathcal{H}$; the magnetic flux density at a point is expressed in lines per square centimeter and is represented by $\mathcal{B}$.

Fig. 37 shows the magnetic fields produced in certain cases. The lines of induction are all closed lines and extend from a north to a south pole in air and from a south to a north pole inside the magnetic material or the generator of m.m.f.



(a) Field of a Bar Magnet

(b) Field of a Solenoid

(c) Field near an Isolated North Pole

(d) Field of a Dynamo

Fig. 37. Magnetic fields.

33. Magnetic Flux. The total number of lines of magnetic induction passing through a given section is called the magnetic flux through the section and is represented by Φ.

The unit of magnetic flux, which is one line, is called the maxwell.

Flux from Unit Pole. At every point on a sphere of one centimeter radius, surrounding a unit pole as centre, a similar unit pole is repelled with a force of one dyne. There must therefore be one line of induction per square centimeter passing through the surface, and since the surface is 4π sq. cm. the total flux from the unit pole is

$$\Phi = 4\pi \text{ lines.}$$

The flux from a pole of strength m is

$$\Phi = 4\pi m \text{ lines.}$$

Thus a unit north pole is associated with each 4π lines leaving a surface and a unit south pole with each 4π lines entering a surface.

34. Magnetic Potential. The magnetic potential of an isolated magnetic pole is the work done in carrying a unit north pole from an infinite distance to the point against the forces in the magnetic field.

Fig. 38 shows a north magnetic pole of strength m. Its field

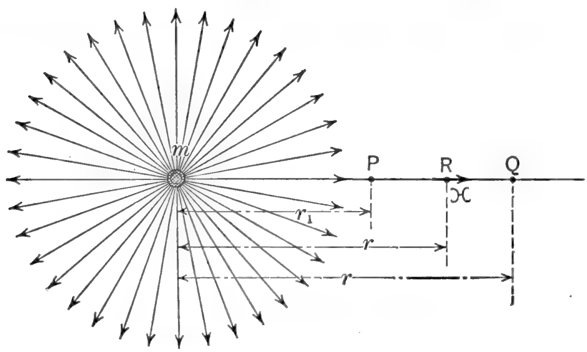


FIG. 38. Magnetic potential.

extends out radially in all directions and the magnetic force at a distance of r cm. from m is

$$\mathcal{H} = \frac{m}{r^2} \text{ dynes.}$$

The magnetic potential of the point P at a distance of r_1 cm. from m is the work done in carrying a unit north pole from an infinite distance to the point against the force of repulsion of m.

The work done is

$$W = \int_{r_1}^{\infty} \mathcal{H}\, dr = \int_{r_1}^{\infty} m \frac{dr}{r^2} = \frac{m}{r_1} \text{ ergs.}$$

Therefore the magnetic potential of a point at a distance of r_1 cm. from an isolated magnetic pole of strength m is

$$M = \frac{m}{r_1}. \qquad \qquad (71)$$

The difference of magnetic potential between the points P and Q in Fig. 38 is

$$M = \int_{r_1}^{r_2} \mathcal{H}\, dr = \int_{r_1}^{r_2} m \frac{dr}{r^2} = \frac{m}{r_1} - \frac{m}{r_2}.$$

Thus the difference of magnetic potential between two points is the line integral of the magnetic force between them.

35. Magnetomotive Force. The difference of magnetic potential or the line integral of the magnetic force between two points is called the magnetomotive force (m.m.f.) between the points. It causes magnetic flux to pass from one to the other.

Unit m.m.f. will produce one line of magnetic flux through a (cm.)^3 of air and μ lines through a (cm.)^3 of a magnetic material of permeability μ. It is called the gilbert.

Since magnetomotive force is the line integral of the magnetic force or

$$M = \int \mathcal{H}\, dr,$$

therefore the magnetic force is the space rate of change of m.m.f. or

$$\mathcal{H} = \frac{dM}{dr}, \qquad \text{(72)}$$

and thus magnetic force is the m.m.f. per centimeter.

36. Permeability. Permeability is the ratio of the magnetic conductivity of a substance to the magnetic conductivity of air and is represented by μ.

Lines of magnetic flux pass through air or any other substance except iron, nickel or cobalt, as easily as they do through a vacuum. The permeability of such substances is for all practical purposes the same and is taken as unity. Iron and its compounds and to a lesser degree nickel and cobalt are found to allow magnetic lines to pass through them much more easily than empty space; that is, a given m.m.f. will produce a much larger flux through a volume of iron than it will through an equal volume of air. The permeability of the iron is therefore greater than that of the air and is expressed by some number greater than unity.

The permeability of magnetic materials is not constant but varies with the induction density as shown in Art. 49.

37. Magnetic Reluctance. The reluctance of a magnetic circuit may be defined as the resistance offered by the circuit to the passage of magnetic flux through it and is represented by $\mathcal{R}$.

When unit m.m.f. is applied across a (cm.)^3 of air one line of magnetic flux is produced crossing from one face to the other, or one line of magnetic induction per square centimeter is produced through a length of one centimeter. The reluctance of a

(cm.)^3 of air is taken as the unit of reluctance since unit m.m.f. produces unit flux through it.

When unit m.m.f. is applied across a (cm.)^3 of a magnetic material of permeability μ, the flux produced is μ lines, or the induction density is μ lines per square centimeter. The reluctance of the path is therefore $\frac{1}{\mu}$ units.

If the length of the path is increased to l cm. the m.m.f. per centimeter and the magnetizing force are reduced in the ratio $1 : l$. The flux density is therefore reduced in the same ratio and becomes

$$\mathfrak{B} = \frac{M}{l}\mu = \frac{\mu}{l} \text{ lines per square centimeter.}$$

If now the section of the path is increased to A sq. cm. the m.m.f. per centimeter and the magnetizing force are not changed, and thus the induction density remains the same, but the flux through the path is increased in proportion to the area; it is

$$\Phi = \mathfrak{B}A = \frac{M}{l}\mu A = \frac{M}{\frac{l}{A\mu}} = \frac{1}{\frac{l}{A\mu}} = \frac{\text{m.m.f.}}{\text{reluctance}}.$$

Thus the reluctance of a path of uniform section is

$$\mathfrak{R} = \frac{l}{A\mu}, \qquad \qquad (73)$$

and is directly proportional to its length and inversely proportional to its sectional area and to the permeability of the material forming it.

The equation connecting the m.m.f. acting on a path, the reluctance of the path and the flux through the path can be written in three ways:

(1) $$\Phi = \frac{M}{\mathfrak{R}}, \qquad \qquad (74)$$

the flux is equal to the m.m.f. divided by the reluctance;

(2) $$M = \Phi\mathfrak{R}, \qquad \qquad (75)$$

the m.m.f. is equal to the flux multiplied by the reluctance;

(3) $$\mathfrak{R} = \frac{M}{\Phi}, \qquad \qquad (76)$$

the reluctance is equal to the m.m.f. divided by the flux.

Assuming the flux to be unity in the last equation the reluctance of the path may be defined as the m.m.f. required to produce unit flux through it.

38. Permeance. The permeance of a magnetic path is the reciprocal of its reluctance and is represented by $\mathfrak{P}$; thus

$$\mathfrak{P} = \frac{1}{\mathfrak{R}} = \frac{\Phi}{M}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

and assuming that the m.m.f. acting is unity, the permeance may be defined as the flux through the path produced by unit m.m.f.

The permeance of a path of uniform section is

$$\mathfrak{P} = \frac{A\mu}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

and is directly proportional to the sectional area and to the permeability, and is inversely proportional to the length of the path.

39. Electromagnetics. The region surrounding a conductor carrying a current of electricity is a magnetic field. A current of electricity therefore represents a magnetomotive force. If the conductor is isolated from other magnetic forces the lines of force will form circles around it.

Maxwell's Corkscrew Rule. The direction of the current and that of the resulting magnetic force are related to one another as the forward travel and the twist of an ordinary corkscrew. This rule is illustrated in Fig. 39.

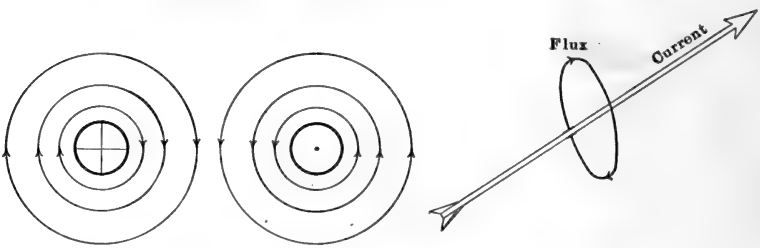


Fig. 39. Magnetic flux produced by electric current.

The symbol $\oplus$ represents a current flowing down and $\odot$ a current flowing up.

Faraday discovered that a current is induced in a closed coil of wire when a magnet is brought near it. The same effect is noticeable if a coil of wire carrying current is moved to or from

the closed coil, or if the second coil is fixed in position and the current in it is varied. The induced current only exists while the magnet or inducing coil is moving with respect to the fixed coil or while the current in the inducing coil is varying.

The induced current is due to the fact that a difference of potential or electromotive force is produced in the circuit by changing the number of lines of magnetic flux threading through it or by causing lines of magnetic flux to cut across it.

40. Laws of Induction. *First Law.* — A change in the number of lines which pass through a closed circuit induces a current around the circuit in such a direction as to oppose the change in the flux threading the circuit.

Second Law. The electromotive force induced around a closed circuit is equal to the rate of change of the flux which passes through the circuit; or the electromotive force induced in a conductor is equal to the rate at which it cuts across lines of magnetic flux.

41. Unit of Electromotive Force. The absolute unit of electromotive force (e.m.f.) is the electromotive force induced in a coil of one turn when the flux threading the coil is changing at the rate of one line per second; or it is the electromotive force induced in a conductor when it is cutting one line per second.

The practical unit is the electromotive force produced by cutting 10^8 lines per second and is called the volt. Electromotive force is commonly called voltage.

To change from absolute units of electromotive force to volts divide by 10^8.

If a coil of wire has n turns and the flux through it is changing at the rate $\frac{d\phi}{dt}$ lines per second the e.m.f. induced in the coil is

$$e = -n\frac{d\phi}{dt} \text{ absolute units.} \qquad (79)$$

The negative sign is used because when the flux is decreasing the induced e.m.f. is in the positive direction, that is, it tends to prevent the decrease of the flux.

42. Force Exerted by a Magnetic Field on an Electric Circuit. Every part of an electric circuit situated in a magnetic field is acted upon by a force tending to move it into the position where it will include the greatest possible flux.

In Fig. 40 (*a*) represents a uniform magnetic field between two unlike poles and (*b*) represents the field surrounding a conductor carrying current. If the conductor is placed in the uniform field the resultant distribution will be as shown in (*c*). The intensity

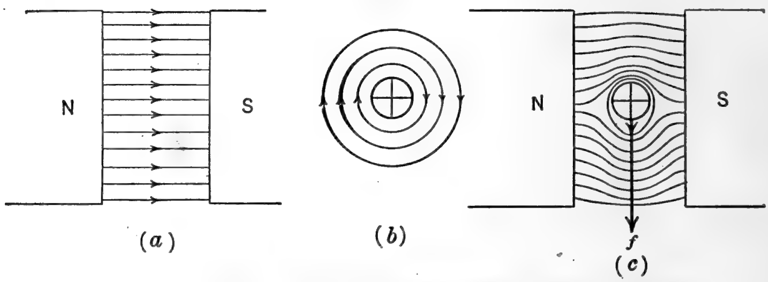


FIG. 40. Force on an electric conductor in a magnetic field.

of the field above the conductor will be greater than that below and a force f will act on the conductor at right angles to it and to the lines of flux, tending to push it out of the field. This force is directly proportional to the intensity of the field or the flux den-

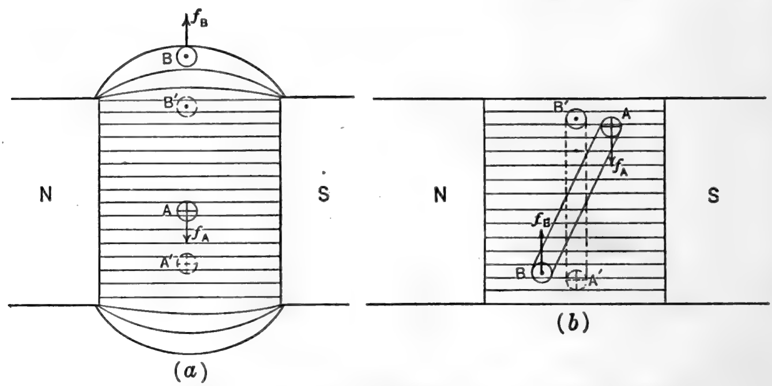


FIG. 41. Force exerted by a magnetic field on a coil of wire.

sity, to the length of the conductor in the field and to the strength of the current, or

$$f = \mathfrak{B} l I \text{ dynes}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

where $\mathfrak{B}$ is the flux density in lines per square centimeter,

l is the length of the conductor in centimeters,

I is the strength of the current in absolute units.

Unit Current. If a conductor carrying one absolute unit of current is placed in a magnetic field of unit strength at right angles to the lines of force, each centimeter of its length will be acted upon by a force of one dyne.

The practical unit of current is one tenth of this absolute unit and is called the ampere.

In Fig. 41 (*a*) AB is a coil of wire carrying current placed in a magnetic field. A is acted upon by a force f_A tending to move it down and B is acted upon by a force f_B tending to move it up. Since A is in the stronger field f_A is greater than f_B and the coil will move down to the position A_1 B_1 where it is inclosing the maximum flux.

Fig. 41 (*b*) shows the direction of the forces acting on the coil in another position. The coil is forced around until it reaches the position A_1 B_1.

43. Transformation of Mechanical Energy to Electrical Energy. In Fig. 42, if the conductor is moved through a distance

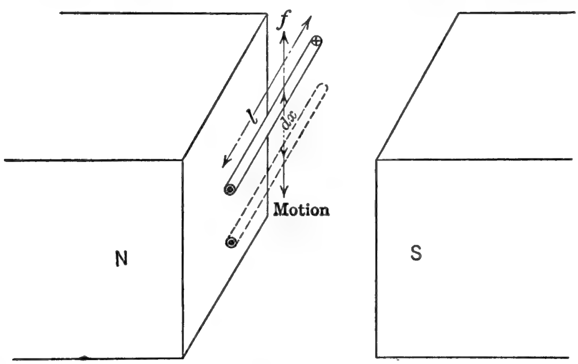


FIG. 42. Transformation of mechanical to electrical energy.

dx at right angles to the flux against the force f, the work done is

$$dw = f\,dx = \mathfrak{B}lI\,dx = \mathfrak{B}l\,dx\,I,$$

but $\mathfrak{B}l\,dx$ is the flux cut in moving through the distance dx and is $= d\phi$, therefore

$$dw = I\,d\phi, \quad \dots \dots \dots \quad (81)$$

and the work done in moving a current across a magnetic field is equal to the product of the current and the flux cut.

In moving completely across the pole face the work done is

$$W = \int I\,d\phi = I\Phi,$$

where Φ is the flux from the pole.

If I is expressed in absolute units and Φ in maxwells, W is in ergs.

If the motion through the distance dx takes place in time dt seconds, the work done is

$$dw = I\,d\phi = I\,dt\frac{d\phi}{dt} = eI\,dt, \quad . \quad . \quad . \quad . \qquad (82)$$

where $e = \frac{d\phi}{dt}$ is the electromotive force generated in the conductor and $I\,dt = dq$ is the quantity of electricity raised through the difference of potential e. Therefore the mechanical work supplied to move the conductor through the distance dx against the force f is used up in doing the electrical work of raising a quantity of electricity dq through a difference of potential e, or in driving a current I against an e.m.f. e for a time dt. Thus mechanical energy is transformed into electrical energy. This is what takes place in an electric generator.

If electric power is supplied to drive the current I against the electromotive force $e = \frac{d\phi}{dt}$, for a time dt energy is supplied

$$dw = e\,I\,dt = \frac{d\phi}{dt}I\,dt = I\,d\phi;$$

the conductor exerts a force $f = \mathfrak{B}l\,I$ dynes through a distance dx and does mechanical work,

$$f\,dx = \mathfrak{B}lI\,dx = I\,d\phi.$$

This is the action of an electric motor.

Electric Power. The electric power in a circuit is the rate at which energy is being transformed in the circuit. It is the product of the current and the electromotive force in the circuit.

$$P = \frac{dw}{dt} = I\frac{d\phi}{dt} = e\,I. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (83)$$

The practical unit of electric power is the watt. It is the power in a circuit carrying one ampere when the electromotive force across it is one volt. The kilowatt, which is one thousand watts, is more commonly used where the amounts of power are large. One horse power is equivalent to 746 watts.

The electric energy transformed in a circuit is the product of the power and the time. The practical units of electric energy are the watt second or joule, the watt hour and the kilowatt hour.

1 watt hour = 3600 watt seconds = 2655.4 ft. lbs. = 3.413 b.t.u. = 0.001341 horse-power hours.

The absolute unit of electric energy is the erg. 1 watt second = 10^7 ergs.

44. Intensity of Magnetic Fields Produced by Electric Currents. The following cases are of special importance: (A) At the centre of a circular loop of wire carrying a current I absolute units. (Fig. 43.)

If $\mathcal{H}$ is the field intensity at O, the centre of the loop, a magnetic pole of strength m placed at this point will be acted upon by a force

$$f = m\mathcal{H} \text{ dynes}$$

in a direction perpendicular to the plane of the coil.

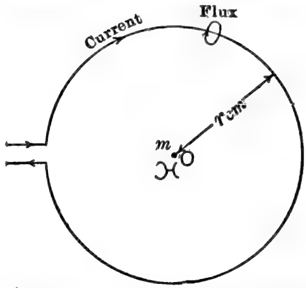


Fig. 43. Intensity of the magnetic field at the centre of a circular coil of wire.

The pole will produce at the wire a field of intensity

$$\mathcal{H}_1 = \frac{m}{r^2} \text{ dynes,}$$

and a flux density

$$\mathcal{B} = \frac{m}{r^2} \text{ lines per square centimeter,}$$

where r cm. is the radius of the loop.

This field will act on the wire with a force

$$f_1 = \mathcal{B}lI \text{ dynes}$$

in a direction perpendicular to the plane of the coil, where $l = 2\pi r$ is the length of the wire in centimeters.

Substituting the values of $\mathcal{B}$ and l gives

$$f_1 = \frac{m}{r^2} \times 2\pi r I = m\frac{2\pi I}{r} \text{ dynes,}$$

but the forces f and f_1 are equal and therefore

$$m\mathcal{H} = m\frac{2\pi I}{r},$$

and

$$\mathcal{H} = \frac{2\pi I}{r} \text{ dynes.} \quad \ldots \ldots \quad (84)$$

The flux density at the centre of the loop is

$$\mathcal{B} = \mathcal{H} = \frac{2\pi I}{r} \text{ lines per square centimeter.}$$

If I is expressed in amperes the field intensity at the centre of the loop is

$$\mathcal{H} = \frac{0.2\pi I}{r} \text{ dynes.} \quad \ldots \ldots \quad (85)$$

(B) At a distance of r cm. from a long straight wire carrying I absolute units. (Fig. 44.)

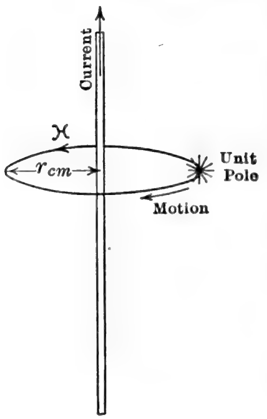


Fig. 44. Intensity of the magnetic field near a long straight wire.

If $\mathcal{H}$ is the intensity of the field, the work done in moving a unit magnetic pole around the wire at a distance r cm. from it against the force $\mathcal{H}$ is

$$w = 2\pi r\mathcal{H} \text{ ergs.}$$

The work done is equal to the product of the current and the flux cut by formula and, therefore,

$$2\pi r\, \mathcal{H} = 4\pi I,$$

and

$$\mathcal{H} = \frac{2I}{r} \text{ dynes;} \quad \ldots \quad (86)$$

the intensity of the field varies directly as the strength of the current and inversely as the distance from the wire.

The flux density at distance r is

$$\mathcal{B} = \mathcal{H} = \frac{2I}{r} \text{ lines per square centimeter.} \quad \ldots \quad (87)$$

If I is in amperes the field intensity is

$$\mathcal{H} = \frac{0.2\,I}{r} \text{ dynes,} \quad \ldots \ldots \quad (88)$$

and the flux density is

$$\mathcal{B} = \frac{0.2\,I}{r} \text{ lines per square centimeter.} \quad \ldots \quad (89)$$

Unit current would produce a flux density of 2 lines per square centimeter at a distance of 1 cm. from a straight wire and it would produce this flux through a distance of 2π cm., or it would produce a flux density of 4π lines per square centimeter through a distance of 1 cm. in air. Thus one absolute unit of current represents a magnetomotive force of 4π gilberts and one ampere represents a magnetomotive force of 0.4π gilberts.

(C) Between two parallel wires *A* and *B*, Fig. 45, at a distance of *D* cm. apart and carrying equal currents *I* but in opposite directions.

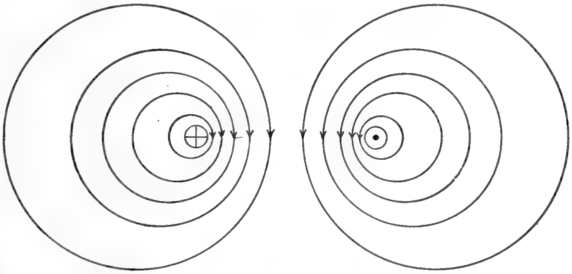


FIG. 45. Magnetic field between two parallel wires.

The field intensity or magnetic force at point *P* distant *r* cm. from *A* and $D-r$ cm. from *B* is the resultant of the magnetic forces due to the currents in *A* and *B*. Since these forces act in the same direction at all points between the wires they can be added directly; the field intensity is by formula

$$\mathcal{H} = \frac{2I}{r} + \frac{2I}{D-r} \text{ dynes}, \quad \ldots \ldots \quad (90)$$

and the flux density at *P* is

$$\mathcal{B} = \mathcal{H} = \frac{2I}{r} + \frac{2I}{D-r} \text{ lines per square centimeter.} \quad (91)$$

(D) At any point on the axis of a short coil of radius *r* cm. (Fig. 46.)

n = number of turns in the coil,
I = current in the coil in absolute units.

Take any point *P* on the axis at a distance *x* cm. from the plane of the coil and let $\mathcal{H}$ be the field intensity there. If a pole of strength *m* is placed at *P* it will be acted on by a force of $m\mathcal{H}$ dynes perpendicular to the plane of the coil. The force exerted

on the coil by the pole is equal and opposite to the force exerted on the pole by the coil.

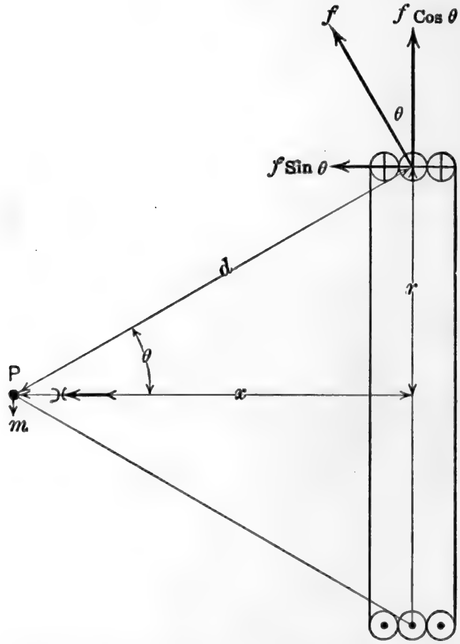


FIG. 46. Intensity of the magnetic field on the axis of a short coil.

The field intensity at the wire due to the pole m at P is $\frac{m}{d^2}$ dynes and the flux density is $\frac{m}{d^2}$ lines per square centimeter, where d cm. is the distance from the point to the wire; the length of wire is $2\,\pi rn$ cm. and therefore the force exerted on it is

$$f = \frac{m}{d^2} \times 2\,\pi rnI \text{ dynes.}$$

This force acts at right angles to the lines of flux and may be resolved into two components, $f \cos\theta$ in the plane of the coil and $f \sin\theta$ perpendicular to the plane of the coil. The component $f \cos\theta$ taken around the loop is zero and therefore the component $f \sin\theta$ is equal in magnitude to the force $m\mathcal{H}$.

The field intensity or magnetic force at P is, therefore,

$$\mathcal{H} = \frac{f \sin \theta}{m} = \frac{2\pi r n I}{d^2} \sin \theta$$

$$= \frac{2\pi r n I}{r^2 + x^2} \times \frac{r}{\sqrt{r^2 + x^2}} = \frac{2\pi r^2 n I}{(r^2 + x^2)^{\frac{3}{2}}} \text{ dynes,} \quad . \quad (92)$$

since $\quad d = \sqrt{r^2 + x^2}$ and $\sin \theta = \dfrac{r}{\sqrt{r^2 + x^2}}$.

(E) On the axis of a long solenoid. In Fig. 47 AB is a solenoid

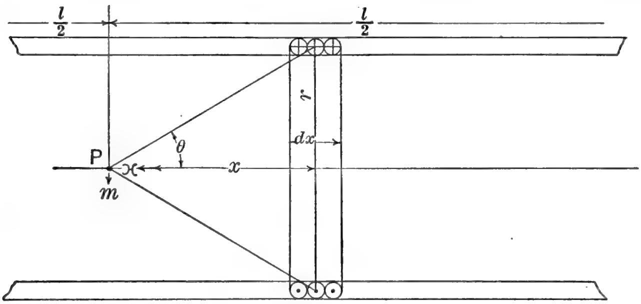


FIG. 47. Magnetic field in a solenoid.

of length l cm. and radius r cm. If n is the number of turns in the solenoid, the number of turns in the section CD of width dx is $\frac{n}{l}\,dx$.

The field intensity at p due to the section CD is by equation 92

$$d\mathcal{H} = \frac{2\pi r^2 I}{(r^2 + x^2)^{\frac{3}{2}}} \times \frac{n}{l}\,dx,$$

where I is the current in the solenoid.

The field intensity due to the complete solenoid is

$$\mathcal{H} = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{2\pi r^2 I}{(r^2 + x^2)^{\frac{3}{2}}} \times \frac{n}{l}\,dx.$$

To integrate this let angle $GPT = \theta$, then

$$x = r \cot \theta,$$

$$(r^2 + x^2)^{\frac{3}{2}} = r^3 (1 + \cot^2 \theta)^{\frac{3}{2}} = r^3 \operatorname{cosec}^3 \theta = \frac{r^3}{\sin^3 \theta},$$

$$dx = -r \operatorname{cosec}^2 \theta \, d\theta = -\frac{r \, d\theta}{\sin^2 \theta};$$

if the solenoid is assumed to be very long the limits of θ may be taken as 0 and π, and, therefore,

$$\mathcal{H} = \frac{2\pi r^2 nI}{l} \int_{\pi}^{0} \frac{\sin^3 \theta}{r^3} \times \left(-\frac{r \, d\theta}{\sin^2 \theta}\right)$$

$$= -\frac{2\pi nI}{l} \int_{\pi}^{0} \sin \theta \, d\theta$$

$$= \frac{2\pi nI}{l} [\cos \theta]_{\pi}^{0}$$

$$= \frac{4\pi nI}{l} \text{ dynes.} \quad \quad (93)$$

If the current is expressed in amperes

$$\mathcal{H} = \frac{0.4\pi nI}{l} \text{ dynes,} \quad \quad (94)$$

and the field intensity on the axis of a long solenoid is proportional to the product of amperes and turns or ampere turns and is inversely proportional to the length of the solenoid.

The field intensity throughout the volume enclosed by the solenoid is practically uniform except near the ends and can be expressed by equation 94.

45. Magnetomotive Force of a Solenoid. The m.m.f. of a solenoid is the line integral of the magnetic forces along any closed path through it and is measured by the work done in carrying a unit magnetic pole around the closed path. (Fig. 48.)

The work done is equal to the product of the current and the flux cut, and thus

m.m.f. $= 4\pi nI$, where I is in absolute units,

or

m.m.f. $= 0.4\pi nI$, where I is in amperes.

The magnetomotive force is proportional to the ampere turns of the coil. It does not make any difference whether it is a small current in a large number of turns or a large current in a small number of turns.

The m.m.f. of one ampere in one turn or one ampere turn is $0.4\,\pi$ gilberts.

In studying the characteristics of electrical machinery it is more convenient to use the ampere turn as the unit of m.m.f.

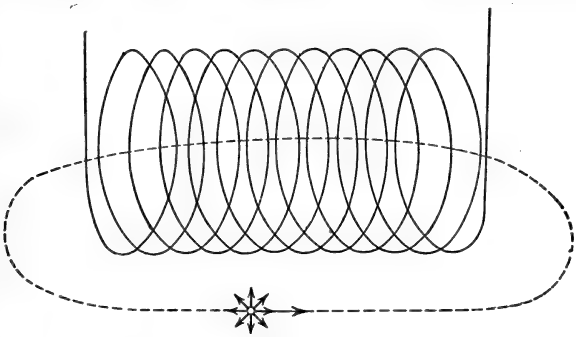


FIG. 48. Magnetomotive force of a solenoid.

Thus the m.m.f. of a field coil of n turns carrying a current I amperes is specified as nI ampere turns instead of $0.4\,\pi nI$ gilberts.

46. Examples. (1) The solenoid in Fig. 49 has nI ampere

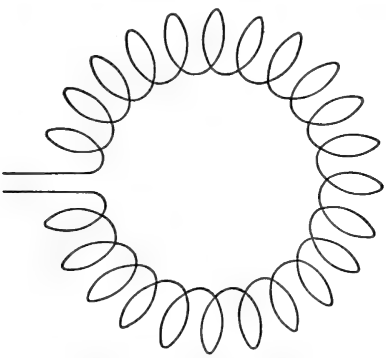


FIG. 49. Ring solenoid.

turns and is wound in the form of a ring. The m.m.f. of the solenoid is $M = 0.4\,\pi nI$ gilberts.

If l cm. is the mean length of the path and A sq. cm. is the sectional area of the path, the reluctance is

$$\Re = \frac{l}{A},$$

and the flux inside the ring is

$$\Phi = \frac{M}{R} = \frac{0.4\,\pi n I}{l/A} \text{ lines.}$$

The flux density in the ring is

$$\mathfrak{B} = \frac{\Phi}{A} \text{ lines per square centimeter,}$$

and the magnetizing force is

$$\mathfrak{H} = \mathfrak{B} = \frac{\Phi}{A} = \frac{0.4\,\pi n I}{l}$$

and is the m.m.f. per centimeter.

If the solenoid is wound on an iron ring of permeability μ, the reluctance is reduced and becomes

$$\mathfrak{R}_1 = \frac{l}{A\mu},$$

the flux is increased to

$$\Phi_1 = \frac{0.4\,\pi n I}{\dfrac{l}{A\mu}}$$

and the flux density or induction density is increased to

$$\mathfrak{B}_1 = \frac{\Phi_1}{A};$$

the magnetizing force remains the same as before,

$$\mathfrak{H} = \frac{\mathfrak{B}_1}{\mu} = \frac{\Phi_1}{A\mu} = \frac{0.4\,\pi n I}{l}.$$

(2) Fig. 50 represents a solenoid of n turns carrying a current of I amperes; the m.m.f. is $M = 0.4\,\pi n I$ gilberts. If l cm. is the length of the solenoid and A sq. cm. is its sectional area, the reluctance of the path through it is

$$\mathfrak{R}_1 = \frac{l}{A}.$$

The m.m.f. required to drive the flux Φ through this reluctance is

$$M_1 = \Phi \frac{l}{A} \text{ gilberts.}$$

But the flux going out at one end has to pass, around through the air and in at the other end as shown. The reluctance of this return path is difficult to calculate and is not of great practical importance; its length is greater than l and its sectional area is very much greater than A and so its reluctance is small compared to $\mathfrak{R}_1$. Thus in the case of long solenoids of small section the

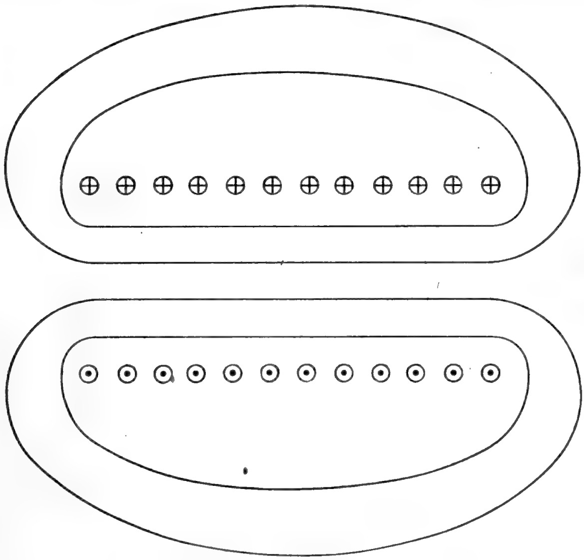


FIG. 50. Solenoid in air.

reluctance of the return path may be neglected and the assumption may be made that the whole m.m.f. M is utilized in driving the flux through the reluctance $\mathfrak{R}_1$. Thus

$$\Phi = \frac{M}{\mathfrak{R}_1} = \frac{0.4\,\pi n I}{l/A},$$

the induction density in the solenoid is

$$\mathfrak{B} = \frac{\Phi}{A} = \frac{0.4\,\pi n I}{l/A}$$

and the magnetizing force at any point inside the solenoid is

$$\mathfrak{H} = \mathfrak{B} = \frac{0.4\,\pi n I}{l/A} \text{ dynes, as in equation 94.}$$

If an iron bar of length l and permeability μ is placed in the solenoid the reluctance of the path through the solenoid is reduced in the ratio $\frac{1}{\mu}$ and the reluctance of the return path is no longer negligible in comparison to it and its value must be calculated. (Fig. 51.)

(3) In Fig. 52 a solenoid of nI ampere turns is wound on an iron ring of permeability μ and a section of length l_2 is cut from the iron. If l_1 is the mean length of the path through the iron and its sectional area is A sq. cm. the reluctance of the path through the iron is

$$\mathfrak{R}_1 = \frac{l_1}{A\mu},$$

the reluctance of the path through the air is

$$\mathfrak{R}_2 = \frac{l_2}{A}$$

and the total reluctance of the path is

$$\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2 = \frac{l_1}{A\mu} + \frac{l_2}{A}.$$

The flux through the path is

$$\Phi = \frac{M}{\mathfrak{R}} = \frac{0.4\,\pi nI}{\frac{l_1}{A\mu} + \frac{l_2}{A}} \quad \ldots\ldots\ldots \quad (95)$$

The flux density in the iron is $\mathfrak{B} = \frac{\Phi}{A}$ and is the same as in the air.

The magnetizing force or m.m.f. per centimeter in the iron is

$$\mathfrak{H}_1 = \frac{\mathfrak{B}}{\mu} = \frac{\Phi}{A\mu},$$

and in the air it is

$$\mathfrak{H}_2 = \mathfrak{B} = \frac{\Phi}{A}.$$

The m.m.f. consumed in the iron is

$$M_1 = \mathfrak{H}_1 l_1 = \frac{\Phi}{A\mu} l_1$$

and the m.m.f. consumed in the air is

$$M_2 = \mathfrak{H}_2 l_2 = \frac{\Phi}{A} l_2.$$

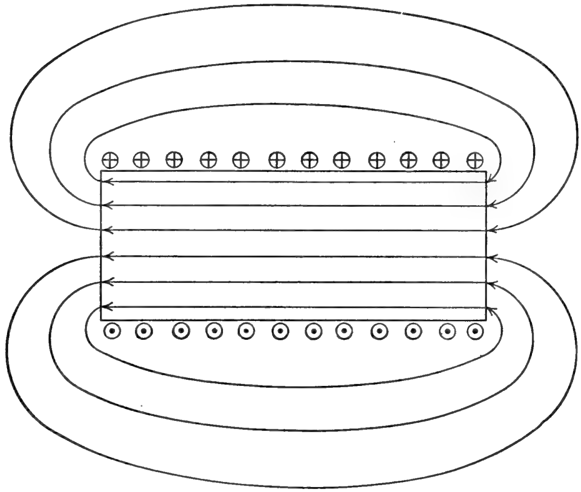


FIG. 51. Solenoid wound on an iron bar.

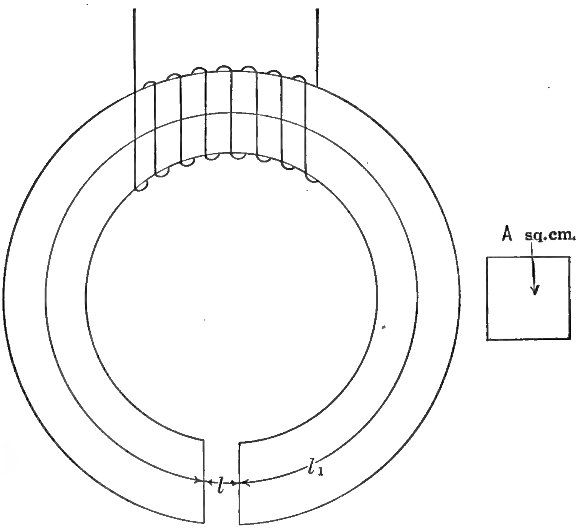


FIG. 52. Ring with an air gap.

(4) Fig. 53 represents the magnetic circuit of a bipolar dynamo. It consists of a number of parts of different materials as follows:

1 yoke y of section A_y, length l_y, and permeability μ_y,
2 pole cores c,c of section A_c, length l_c, and permeability μ_c,
2 pole pieces p,p of section A_p, length l_p, and permeability μ_p,
2 air gaps g,g of section A_g, length l_g, and permeability l,
1 armature a of section A_a, length l_a, and permeability μ_a.

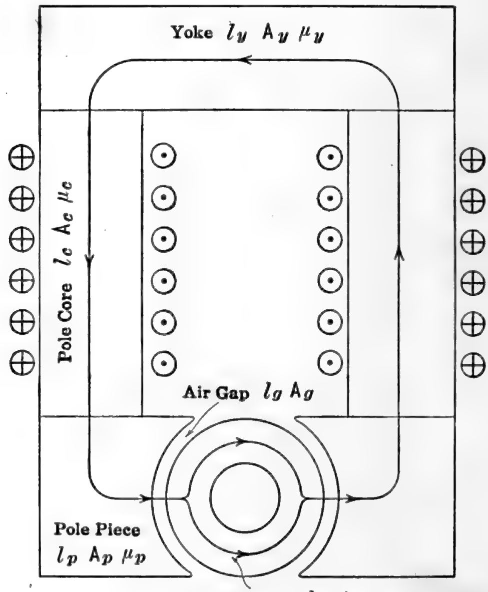


FIG. 53. Magnetic circuit of a dynamo.

The reluctance of these parts are, respectively,

$$\mathfrak{R}_y = \frac{l_y}{A_y\mu_y}, \quad \mathfrak{R}_c = \frac{2\,l_c}{A_c\mu_c}, \quad \mathfrak{R}_p = \frac{2\,l_p}{A_p\mu_p}, \quad \mathfrak{R}_g = \frac{2\,l_g}{A_g}, \quad \mathfrak{R}_a = \frac{l_a}{A_a\mu_a};$$

and the reluctance of the whole circuit is

$$\mathfrak{R} = \mathfrak{R}_y + \mathfrak{R}_c + \mathfrak{R}_p + \mathfrak{R}_g + \mathfrak{R}_a.$$

The m.m.f. M is provided by field coils placed on the pole cores as shown, and the flux through the circuit is

$$\Phi = \frac{M}{\mathfrak{R}} = \frac{M}{\mathfrak{R}_y + \mathfrak{R}_c + \mathfrak{R}_p + \mathfrak{R}_g + \mathfrak{R}_a},$$

and is equal to the m.m.f. divided by the total reluctance; and the m.m.f. is

$$M = \Phi\mathfrak{R} = \Phi\mathfrak{R}_y + \Phi\mathfrak{R}_c + \Phi\mathfrak{R}_p + \Phi\mathfrak{R}_g + \Phi\mathfrak{R}_a$$
$$= M_y + M_c + M_p + M_g + M_a,$$

where M_y is the part of the total field m.m.f. required to drive the flux through the yoke, etc.

The m.m.f. M_g required to drive the flux across the air gaps is sometimes as much as 80 per cent of the total m.m.f.

(5) Determine the reluctance of the ring in Fig. 54 made up of

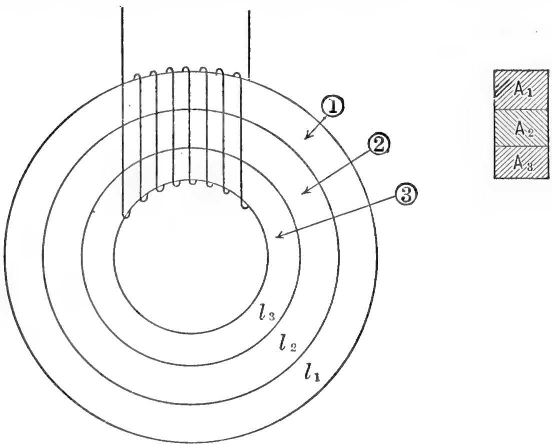


FIG. 54.

three parts of lengths l_1, l_2 and l_3 cm. respectively and sectional areas A_1, A_2 and A_3 sq. cm. and permeabilities μ_1, μ_2 and μ_3. The m.m.f. of the solenoid is M.

The reluctance of section (1) is $\mathfrak{R}_1 = \dfrac{l_1}{A_1\mu_1}$;

the reluctance of section (2) is $\mathfrak{R}_2 = \dfrac{l_2}{A_2\mu_2}$;

the reluctance of section (3) is $\mathfrak{R}_3 = \dfrac{l_3}{A_3\mu_3}$;

the flux through section (1) is $\Phi_1 = \dfrac{M}{\mathfrak{R}_1}$;

the flux through section (2) is $\Phi_2 = \dfrac{M}{\mathfrak{R}_2}$;

the flux through section (3) is $\Phi_3 = \dfrac{M}{\mathfrak{R}_3}$;

the total flux through the ring is

$$\begin{aligned}\Phi &= \Phi_1 + \Phi_2 + \Phi_3 \\ &= \frac{M}{\mathfrak{R}_1} + \frac{M}{\mathfrak{R}_2} + \frac{M}{\mathfrak{R}_3} \\ &= M\left(\frac{1}{\mathfrak{R}_1} + \frac{1}{\mathfrak{R}_2} + \frac{1}{\mathfrak{R}_3}\right). \qquad (96)\end{aligned}$$

But the flux is equal to the m.m.f. divided by the reluctance of the ring, or

$$\Phi = \frac{M}{\mathfrak{R}},$$

and therefore

$$\mathfrak{R} = \frac{M}{\Phi} = \frac{1}{\dfrac{1}{\mathfrak{R}_1} + \dfrac{1}{\mathfrak{R}_2} + \dfrac{1}{\mathfrak{R}_3}}. \qquad (97)$$

(6) Fig. 55 shows a ring of iron with a piece set in made up of three parts of different permeabilities. The lengths, sections and

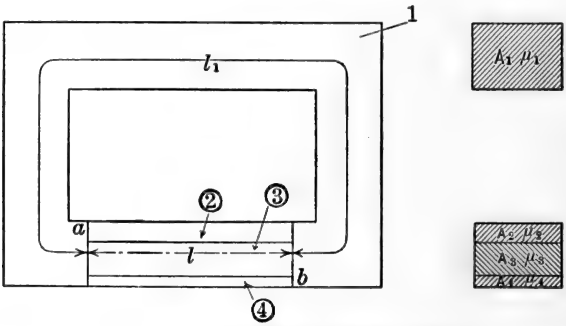


FIG. 55. Series parallel magnetic circuit.

permeabilities are indicated and the reluctances of the various parts are $\mathfrak{R}_1 = \dfrac{l_1}{A_1\mu_1}$, $\mathfrak{R}_2 = \dfrac{l}{A_2\mu_2}$, $\mathfrak{R}_3 = \dfrac{l}{A_3\mu_3}$, and $\mathfrak{R}_4 = \dfrac{l}{A_4\mu_4}$. Determine the reluctance of the circuit.

Let M = the m.m.f. applied to the ring,
Φ = flux through the ring,
M_1 = m.m.f. consumed in section (1),
M_2 = m.m.f. consumed in section ab,

then $M_1 = \Phi\mathfrak{R}_1$, and $M_2 = \Phi\mathfrak{R}_{ab}$.

The reluctance of the section *ab* consisting of three paths in multiple must be found. The flux Φ passing through it divides into three parts which are inversely proportional to the reluctances of the paths,

$$\Phi_2 = \frac{M_2}{\mathfrak{R}_2}, \quad \Phi_3 = \frac{M_2}{\mathfrak{R}_3}, \quad \Phi_4 = \frac{M_2}{\mathfrak{R}_4},$$

therefore,

$$\Phi = \Phi_2 + \Phi_3 + \Phi_4 = M_2\left(\frac{1}{\mathfrak{R}_2} + \frac{1}{\mathfrak{R}_3} + \frac{1}{\mathfrak{R}_4}\right) = \frac{M_2}{\mathfrak{R}_{ab}},$$

and

$$\mathfrak{R}_{ab} = \frac{1}{\frac{1}{\mathfrak{R}_2} + \frac{1}{\mathfrak{R}_3} + \frac{1}{\mathfrak{R}_4}}.$$

The reluctance of the whole circuit is

$$\mathfrak{R} = \frac{M}{\Phi} = \frac{M_1 + M_2}{\Phi} = \mathfrak{R}_1 + \mathfrak{R}_{ab}$$

$$= \mathfrak{R}_1 + \frac{1}{\frac{1}{\mathfrak{R}_2} + \frac{1}{\mathfrak{R}_3} + \frac{1}{\mathfrak{R}_4}}.$$

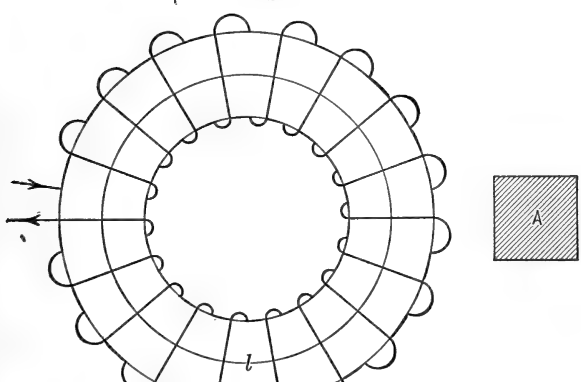


FIG. 56. Energy stored in a magnetic field.

47. Energy Stored in the Magnetic Field. When a current i c.g.s. units flows in the solenoid, Fig. 56, wound on an iron ring, a flux is produced,

$$\Phi = \frac{4\,\pi n i}{l} A\mu,$$

where n is the number of turns in the solenoid,
A is the sectional area of the ring,
l is the mean length of the ring and
μ is its permeability.

When the current increases by a small amount di the flux increases by an amount $d\phi = \frac{4\pi n A\mu}{l} di$, and the work done is equal to the product of the current and the increase of the flux,

$$d\omega = ni\, d\phi = \frac{4\pi n^2 A\mu}{l} i\, di \text{ ergs.}$$

The work done while the current is building up to its full value I is

$$W = \int d\omega = \frac{4\pi n^2 A\mu}{l} \int_0^I i\, di$$

$$= \frac{4\pi n^2 A\mu}{l} \frac{I^2}{2} \text{ ergs} \quad \ldots \ldots \ldots \quad (98)$$

$$= L \frac{I^2}{2} \text{ ergs,} \quad \ldots \ldots \ldots \quad (99)$$

where $L = \frac{4\pi n^2 A\mu}{l}$ is a constant and is called the inductance of the circuit.

This amount of energy is stored in the magnetic field of the solenoid and may be expressed in other forms.

The energy stored is

$$W = \frac{4\pi n^2 A\mu}{l} \frac{I^2}{2} \text{ ergs,}$$

but the magnetic force in the solenoid is

$$\mathcal{H} = \frac{4\pi n I}{l},$$

therefore,

$$W = \frac{\mu \mathcal{H}^2}{8\pi} Al,$$

or since the induction density is

$$\mathcal{B} = \mu \mathcal{H},$$

the energy may be expressed as

$$W = \frac{\mathcal{B}^2}{8\pi\mu} Al.$$

The product Al represents the volume of the magnetic field and therefore the energy stored in the field per unit volume is

$$\omega = \frac{W}{Al} = \frac{\mu \mathfrak{H}^2}{8\pi}, \quad \dots \dots \dots \quad (100)$$

or

$$\omega = \frac{\mathfrak{B}^2}{8\pi\mu}, \quad \dots \dots \dots \quad (101)$$

or

$$\omega = \frac{\mathfrak{H}\mathfrak{B}}{8\pi} \text{ ergs.} \quad \dots \dots \dots \quad (102)$$

When the magnetic field is produced in air the field intensity $\mathfrak{H}$ is equal to the flux density $\mathfrak{B}$, and the energy stored per cubic centimeter is

$$\omega = \frac{\mathfrak{H}^2}{8\pi} \text{ ergs,} \quad \dots \dots \dots \quad (103)$$

or

$$\omega = \frac{\mathfrak{B}^2}{8\pi} \text{ ergs.} \quad \dots \dots \dots \quad (104)$$

48. Stress in the Magnetic Field. The force between two magnetic poles is not exerted at a distance but is transmitted through the medium separating them and the medium is stressed.

There is a tension along the lines of force or induction tending to shorten them and draw the boundary surfaces of the field together and so decrease the magnetic energy stored in the field; there is also a pressure at right angles to the lines tending to spread them apart, and so decrease the flux density in the field and, therefore, also the energy stored. Since the medium is in equilibrium these two forces are equal in magnitude.

The pull on the bounding surfaces is usually expressed in dynes per square centimeter and may be found as follows:

The energy stored in a magnetic field in air was found to be $\frac{\mathfrak{B}^2}{8\pi}$ ergs in Art. 47.

Fig. 57 represents a horseshoe magnet with its armature removed a distance x. If $\mathfrak{B}$ is the flux density in the field between the magnet poles and the armature and A sq. cm. is the area of the two pole faces, the volume of the field is Ax cu. cm. and the energy stored in it is

$$W = \frac{\mathfrak{B}^2}{8\pi} Ax \text{ ergs.}$$

If a pull P is exerted on the armature and it is moved through a distance dx, the work done is $P\,dx$ and this is equal to the increase in the energy stored in the field; therefore,

$$P\,dx = \frac{\mathfrak{B}^2}{8\pi} A\,dx$$

and

$$P = \frac{\mathfrak{B}^2 A}{8\pi}; \quad . \quad . \quad (105)$$

this is the pull of the magnet on the armature.

The pull per square centimeter is

$$p = \frac{P}{A} = \frac{\mathfrak{B}^2}{8\pi}, \quad . \quad (106)$$

and this is the tension along the lines of induction in the field and is also equal in magnitude to the pressure at right angles to the lines.

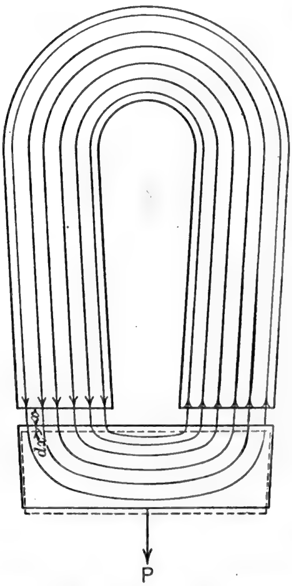


Fig. 57. Pull of a magnet.

49. Permeability Curves. If an iron ring, Fig. 56, which has been completely demagnetized, is gradually magnetized, by increasing the current in the exciting coil from zero, the magnetic induction in the ring increases with the magnetizing force as shown in curve 1, Fig. 58. With feeble magnetizing

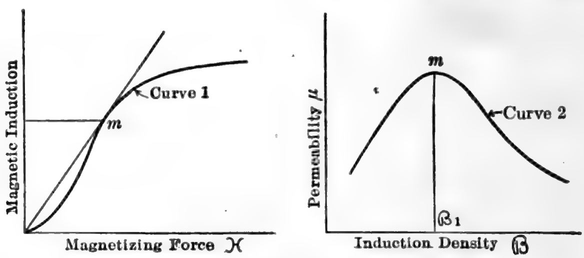


Fig. 58. Permeability curves.

forces the gradient of the curve is small and the permeability is low; as the force increases the curve becomes very steep and nearly straight and the permeability increases rapidly and becomes very large; as the force is further increased the curve rounds off and

the gradient becomes small again and a large increase in magnetizing force is required to produce any considerable increase in density, the permeability decreases to a low value and the material is said to be saturated. Such a curve is called a "permeability curve," "magnetization curve" or "saturation curve" of the material.

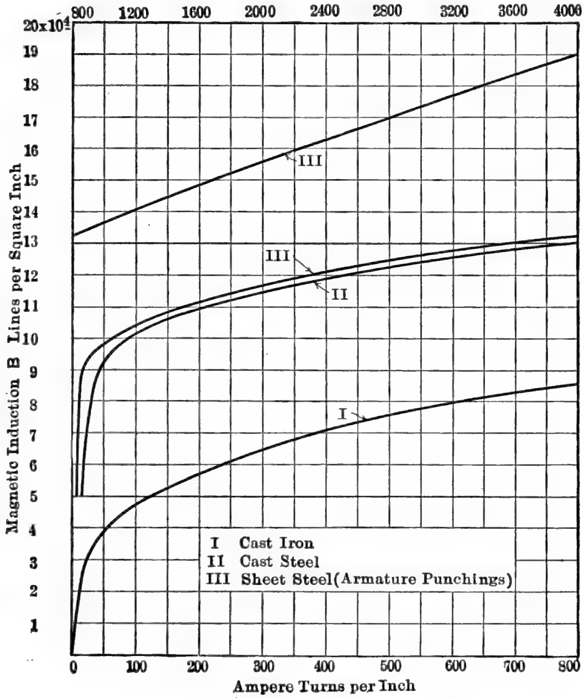


FIG. 59. Magnetization curves.

Curve 2, Fig. 58, shows the relation between the permeability $\mu = \frac{\mathcal{B}}{\mathcal{H}}$ and the induction density $\mathcal{B}$; when the density is low the permeability is low; as the density is increased the permeability increases until it reaches a maximum at the point m where the tangent from the origin touches curve 1; above this point the permeability decreases again.

In Fig. 59 are shown permeability curves for materials used in electrical machine design. For cast iron the maximum value of

μ occurs at a density of about 4000 lines per square centimeter or 26,000 lines per square inch, and for steel at about 6500 lines per square centimeter or 42,000 lines per square inch. Densities below these points are not of great importance. These curves are plotted with induction density B expressed in lines per square inch on a base of ampere turns per inch instead of magnetizing force $\mathcal{H}$. This results in a change of scales only since B lines per square inch $= (2.54)^2\, \mathcal{B}$, and T ampere turns per inch corresponds to a magnetizing force $\mathcal{H} = \frac{0.4\,\pi T}{2.54}$.

The curves showing the relation between permeability μ and induction density B in lines per square inch for the same materials are shown in Fig. 60.

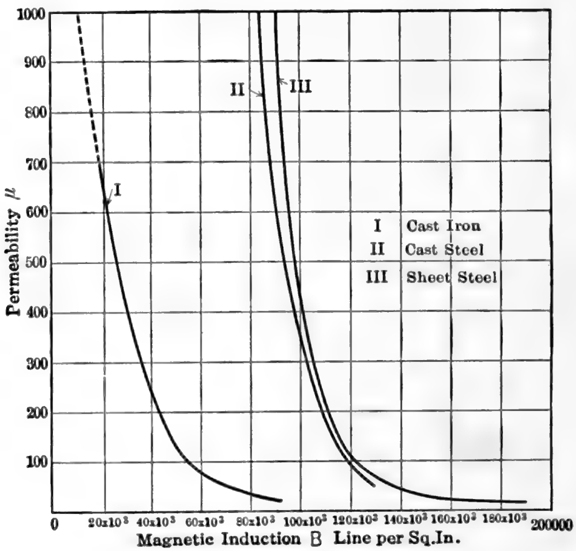


FIG. 60. Permeability curves.

50. Hysteresis. If the exciting current in the solenoid in Art. 47 after being increased to its maximum value I is gradually decreased again, the flux density decreases very slowly and when the current i and therefore the magnetizing force $\mathcal{H}$ is zero it has a value or, Fig. 61, which is called the residual magnetic induction. If now the current is reversed and increased the residual induction is decreased and reaches zero when the magnetizing

force is oc; this force is called the coercive force. If the current is still increased until it reaches its maximum value I again, the induction density will have the same maximum value as before. If the cycle is completed by again reducing and reversing the current the closed curve $arca_1r_1c_1a$ shows the relation between $\mathfrak{B}$ and $\mathfrak{H}$. This closed curve is called a hysteresis loop.

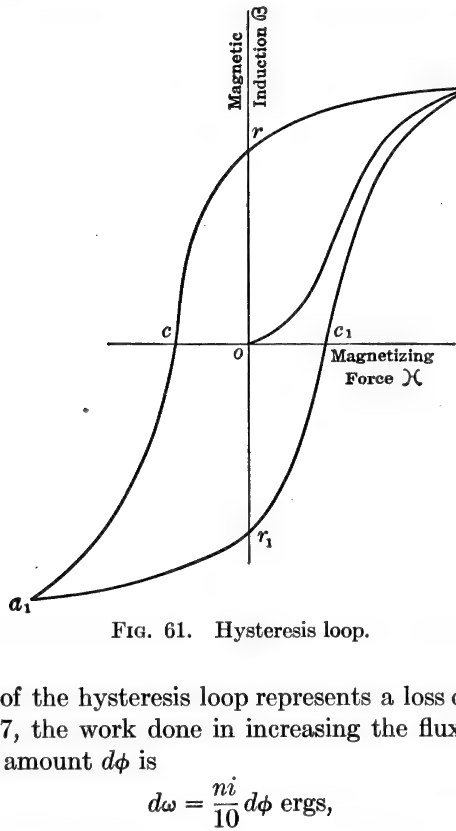


FIG. 61. Hysteresis loop.

The area of the hysteresis loop represents a loss of energy since, from Art. 47, the work done in increasing the flux threading the solenoid by amount $d\phi$ is

$$d\omega = \frac{ni}{10} d\phi \text{ ergs,}$$

where n is the number of turns on the coil and i is the current in amperes flowing when the flux threading the coil is ϕ. If the ring has a constant section A sq. cm. and length l cm. then, since $\mathfrak{H} = \frac{4\pi ni}{10\,l}$ and $d\phi = A\,dB$, the work done is

$$d\omega = \frac{Al}{4\pi} \mathfrak{H}\, d\mathfrak{B}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

and the work done during a complete cycle is

$$W = \int d\omega = \frac{Al}{4\pi}\int \mathfrak{H}\, d\mathfrak{B} \text{ ergs.}$$

Al is the volume of the ring in cubic centimeters and therefore the hysteresis loss in ergs per cycle per cubic centimeter is

$$\omega_h = \frac{1}{4\pi}\int \mathfrak{H}\, d\mathfrak{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

and is thus proportional to the area of the hysteresis loop.

The area of the hysteresis loop and thus the hysteresis loss per cycle increases faster than the maximum induction density. Steinmetz gives the following equation for the loss per cycle per cubic centimeter in terms of the maximum induction,

$$\omega_h = \eta \mathfrak{B}^{1.6} \text{ ergs,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

where η is called the hysteretic constant of the material. For average armature iron the constant has a value about 0.003.

CHAPTER III

ELECTRIC CIRCUITS

51. Ohm's Law. When an electromotive force is applied to the terminals of a conductor, a current is produced which is directly proportional to the e.m.f. and is inversely proportional to the resistance of the conductor;

$$I = \frac{E}{R} \text{ amperes}, \quad \ldots \ldots \quad (110)$$

where I is the current in amperes,
E is the e.m.f. in volts,
R is the resistance in ohms.

This is Ohm's Law.

A conductor has a resistance of one ohm, when an e.m.f. of one volt is required to drive a current of one ampere through it.

When therefore a current I flows through a resistance R, electromotive force is consumed;

$$E = IR \text{ volts}. \quad \ldots \ldots \quad (111)$$

52. Joule's Law. Whenever a current flows through a resistance, electric energy is transformed into heat energy. The power or the rate at which energy is transformed in the circuit is equal to the product of the current and the electromotive force consumed in driving the current through the resistance of the circuit.

$$P = EI \text{ watts},$$

but

$$E = IR \text{ and, therefore,}$$

$$P = I^2R \text{ watts}; \quad \ldots \ldots \quad (112)$$

thus, the power consumed in the circuit is equal to the square of the current multiplied by the resistance. This is Joule's Law.

The power consumed in the resistance of circuits represents a loss of power except in such cases as the incandescent lamp, where it is utilized in producing light, or the electric heater, where the heat developed is applied to a useful purpose.

53. Resistance. The resistance of a conductor varies directly as its length and inversely as its sectional area; it also depends on the material of which the conductor is made;

$$R = \rho \frac{l}{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

where l is the length of the conductor,
A is the sectional area,
ρ is the specific resistance or resistivity of the material.

Wires are usually specified by gauge numbers, their lengths are given in feet and their sectional areas in circular mils. A circular mil is the area of a circle one mil or one thousandth of an inch in diameter. The specific resistance is then the resistance of a wire one foot long and one circular mil in section.

54. Conductance. The reciprocal of the resistance of a conductor is called its conductance and is represented by the letter G, where

$$G = \frac{1}{R}.$$

The reciprocal of the specific resistance or resistivity of a material is called its conductivity and is represented by the Greek letter γ, where

$$\gamma = \frac{1}{\rho}.$$

Conductivity is expressed in per cent of Matthiessen's standard of conductivity. Electrolytic copper sometimes reaches a value of 101 per cent of this standard, but commercial copper ranges from 97 to 99 per cent conductivity. Aluminum wire has a conductivity of 60 or 61 per cent.

55. Effect of Temperature on Resistance. The electric resistance of materials varies with their temperature. The variation of the resistance of metal conductors can be expressed by the following formula,

$$R_t = R_0 (1 + \alpha t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

where R_0 is the resistance at a chosen standard temperature, R_t is the resistance at a temperature t degrees higher and α is the temperature coefficient of resistance. It is the increase in resistance per degree rise in temperature expressed as a fraction of the resistance at the standard temperature. If the centigrade

scale is used and R_0 is the resistance at 0° C., the value of α for copper is 0.00428. This value can be considered as constant over the range of temperatures from 0° C. to 100° C. With the Fahrenheit scale of temperature and R_0 taken as the resistance at 32° F., the value of α is $\frac{0.00428}{1.8} = 0.00249$. If the temperature t_1° C. is chosen as standard the formula can be written

$$R_t = R_{t_1}\{1 + \alpha(t - t_1)\}, \quad \ldots \ldots \quad (115)$$

where α is not the same as before but is smaller because the increase of resistance is expressed as a fraction of the resistance at t_1° C. which is larger than the resistance at 0° C. When t_1 degree is taken as the standard room temperature of 25° C. the value of α is 0.00386.

If the resistance at t_1° C. is known, but the corresponding value of α is not known, the resistance at temperature t can be found from the formula

$$R_t = R_{t_1}\left(\frac{1 + 0.00428\,t}{1 + 0.00428\,t_1}\right). \quad \ldots \ldots \quad (116)$$

This is derived by eliminating R_0 from the two equations,

$$R_t = R_0(1 + 0.00428\,t),$$

and

$$R_{t_1} = R_0(1 + 0.00428\,t_1).$$

Example. If the resistance of a copper conductor at 25° C. is 10 ohms, determine its resistance at 65° C.

From formula 115 the resistance is

$$R_{65} = 10\{1 + 0.00386(65 - 25)\} = 11.55 \text{ ohms};$$

or from formula 116

$$R_{65} = 10\left(\frac{1 + 0.00428 \times 65}{1 + 0.00428 \times 25}\right) = 11.55 \text{ ohms}.$$

The temperature coefficients for all pure non-magnetic metals are practically the same as for copper and the formulæ above may be used. For alloys the temperature coefficient varies from the values given for pure metals to zero. In certain compositions it has a value approximately zero over the range at temperatures met in ordinary practice. Such alloys are very useful in the manufacture of standard resistances, as ammeter shunts, etc., where the resistance must remain constant.

Iron has a temperature coefficient of resistivity of 0.006 per degree centigrade.

Carbon, unlike the metals, has a negative temperature coefficient and it is not constant. The resistance of insulating materials decreases very rapidly with increase of temperature, but the variation is irregular and cannot be expressed by a simple

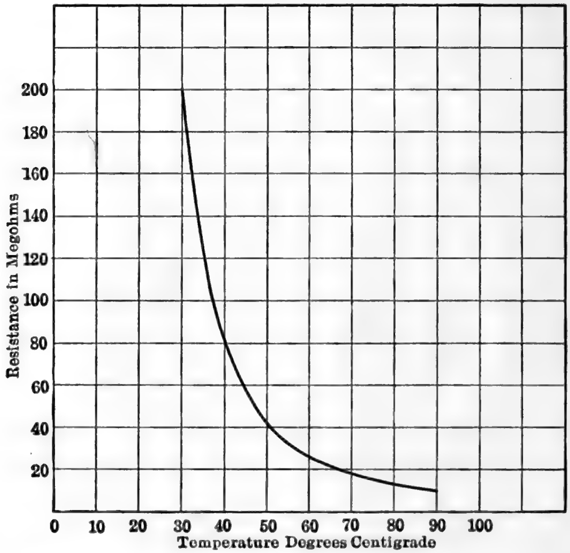


FIG. 62. Variation of resistance of slot insulation with temperature.

equation. Fig. 62 shows the variation of the resistance of an insulating material with temperature. The resistance is expressed in megohms or millions of ohms.

56. Resistance of Conductors. The resistance of a circular mil foot of pure copper wire at 25° C. is 10.55 ohms; therefore, the resistance of a copper wire at 25° C. is, by formula 113,

$$R = 10.55 \frac{\text{length in feet}}{\text{section in circular mils}} = 10.55 \frac{l'}{\text{cir. mils}} \cdot \quad (117)$$

In the case of rectangular conductors, as bus bars, the section is expressed in square mils and the value of ρ is then the resistance of a square wire, one mil on each side and one foot long.

The following table gives the specific resistance of copper at different temperatures and the corresponding values of the temperature coefficient.

Temperature in degrees cent.	Resistance of one circular mil ft. in ohms	Resistance of one sq. mil ft. in ohms	Temperature, coefficient α
0°	9.6	7.55	0.00428
25°	10.55	8.3	0.00386
50°	11.7	9.12	0.00352

In this table and in wire tables generally the conductivity is assumed as 100 per cent. If the conductivity of a given wire is less than this the specific resistance must be changed accordingly; thus the resistance of one circular mil foot of copper of 98 per cent conductivity at 25° C. is $\frac{10.55}{0.98} = 10.76$ ohms; the resistance of one circular mil foot of aluminium of 60 per cent conductivity at 25° C. is $\frac{10.55}{0.60} = 17.58$ ohms.

WIRE TABLE

Gauge number, Brown & Sharpe	Diameter in inches	Area c.m.	Resistance per 1000 ft. at 25° C.	Current capacity, rubber covered	Current capacity, other insulation
0000	0.460	211,600	0.0499	210	312
000	0.4096	167,800	0.0629	177	262
00	0.3648	133,100	0.0794	150	220
0	0.3249	105,500	0.1000	127	185
1	0.2893	83,690	0.1261	107	156
2	0.2576	66,370	0.1591	90	131
3	0.2294	52,630	0.2006	76	110
4	0.2043	41,740	0.2529	65	92
5	0.1819	33,100	0.3190	54	77
6	0.1620	26,250	0.4022	46	65
8	0.1340	17,960	0.6396	33	46
10	0.1019	10,380	1.0171	24	32
12	0.08081	6,530	1.6170	17	23
14	0.06408	4,107	2.571	12	16
16	0.05082	2,583	4.089	6	8
18	0.04030	1,624	6.501	3	5

57. Drop of Voltage and Loss of Power in a Distributing Circuit. The distributing circuit in Fig. 63 delivers 20 kilowatts at 220 volts to a receiver circuit 1000 feet distant; if the size of the conductors is No. 1 B. & S. determine the voltage required at the generating end of the line and the power lost in the line.

E_g is the generator voltage,
E is the receiver voltage = 220 volts,
I is the load current,
r is the resistance of each conductor.

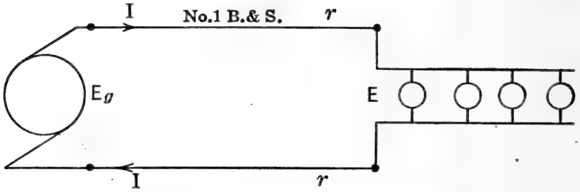


Fig. 63. Distributing circuit.

The power delivered is

$$EI = 20{,}000 \text{ watts};$$

the current is therefore

$$I = \frac{20{,}000}{220} = 90.9 \text{ amperes};$$

the resistance of each conductor at 25° C. is

$$r = 0.126 \text{ ohms};$$

the voltage drop in each conductor is

$$e_1 = Ir = 90.9 \times 0.126 = 11.48 \text{ volts};$$

the generator voltage is therefore

$$E_g = E + 2\,Ir = 220 + 22.96 = 242.96 \text{ volts};$$

the drop of voltage in the circuit is

$$2\,e_1 = 2\,Ir = 22.96 \text{ volts}$$

$$= \frac{22.96}{242.96} \times 100 \text{ per cent} = 9.45 \text{ per cent.}$$

The loss of power in the circuit is

$$2\,I^2r = 2 \times 90.9^2 \times 0.1261 = 2090 \text{ watts};$$

the power delivered by the generator is

$$E_gI = 242.96 \times 90.9 = 22{,}100 \text{ watts};$$

therefore the power loss is

$$\frac{2090}{22{,}100} \times 100 \text{ per cent} = 9.45 \text{ per cent};$$

and the efficiency of the transmission is 90.55 per cent.

If the drop of voltage had been limited to 10 volts what size of wire would have been required?

The drop in voltage is

$$2\,Ir = 2 \times 90.9 \times 10.55\,\frac{1000}{\text{c.m.}} = 10 \text{ volts};$$

therefore the required section in circular mils is

$$A = 2 \times 90.9 \times 10.55\,\frac{1000}{10} = 192{,}000 \text{ c.m.}$$

58. Current-carrying Capacity of Wires. The energy consumed in the resistance of a wire raises the temperature of the wire until the point is reached where the heat radiated and conducted from the wire is equal to the heat generated in it. When the wire is bare the heat will escape easily into the air, but when it is covered with insulating material the heat cannot escape so easily and for a given current density the temperature rise will be greater. This increase in temperature decreases the resistance of the insulating material and so decreases its insulating properties; in extreme cases the insulation may be charred and rendered useless. The last two columns of the table in article 56 give the values of current which can be carried safely by different sizes of wire. With rubber insulation the allowable current is about 25 per cent less than with weatherproof insulation because the rubber is more easily affected by heat. When the insulated wires are inclosed in conduits the current-carrying capacity is less than that given in the table.

59. Examples. (1) Determine the resistance of a copper wire of 97 per cent conductivity, 100,000 c.m. in section and 50 ft. in length at 50° C.

The resistance is

$$R = \frac{11.7}{0.97}\,\frac{50}{100{,}000} = 0.00605 \text{ ohms.}$$

(2) Determine the resistance of an aluminum bar 0.75 in. $\times$ 0.375 in. $\times$ 100 ft. at 25° C., if the conductivity is 60 per cent.

The resistance is

$$R = \frac{8.3}{0.60}\,\frac{100 \text{ ft.}}{750 \times 375} = 0.00492 \text{ ohms.}$$

(3) If the resistance of the shunt-field winding of a generator

is 30 ohms at a room temperature of 25° C. and after running under load is found to be 31.5 ohms, determine the average temperature of the winding.

If t is the average temperature of the winding when hot, formula 115 gives

$$R_t = R_{25}\{1 + 0.00386\,(t - 25)\},$$

or substituting

$$31.5 = 30\{1 + 0.00386\,(t - 25)\};$$

the rise in temperature is

$$t - 25 = \frac{\frac{31.5}{30} - 1}{0.00386} = 13^\circ \text{ C.};$$

and the average temperature of the winding is

$$t = 25 + 13 = 38^\circ \text{ C.}$$

This method is used in measuring the temperature rise in the field and armature windings of electrical machines to replace the inaccurate indications given by the use of thermometers.

60. Kirchoff's Laws. Two laws enunciated by Kirchoff are of great value in solving problems dealing with continuous current circuits.

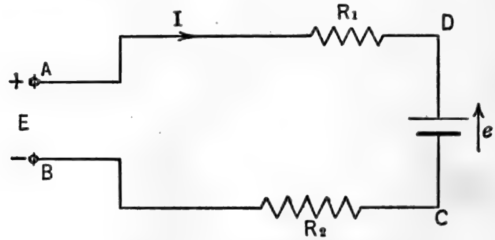


FIG. 64. Kirchoff's first law.

First law. The sum of all the electromotive forces around a closed circuit, taken in their proper direction, is equal to zero; here the e.m.f. consumed by resistance is considered as a counter e.m.f. opposing the current.

Second law. The algebraic sum of all currents flowing toward a distributing point is equal to zero.

The first law is illustrated by the circuit $ABCD$ in Fig. 64. Between the points A and B an e.m.f. E is applied which drives a current I around the circuit in the clockwise direction against the

e.m.f. e of the battery between C and D. R_1 and R_2 are resistances connected in the circuit.

The electromotive force acting in the direction of the current is $E - e$ and is consumed in driving the current I through the two resistances R_1 and R_2 and by Ohm's law.

$$E - e = IR_1 + IR_2.$$

If IR_1 and IR_2 are considered as e.m.f.'s opposing the current,

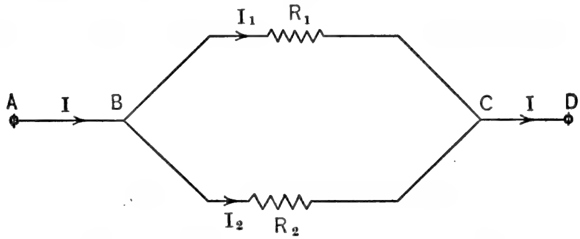


FIG. 65. Kirchoff's second law.

then the sum of all the e.m.f.'s around the closed circuit $ABCD$ is

$$E - e - IR_1 - IR_2 = 0.$$

The second law is illustrated by the circuit $ABCD$ in Fig. 65. The current I divides at B into two parts I_1 and I_2; I is numeri-

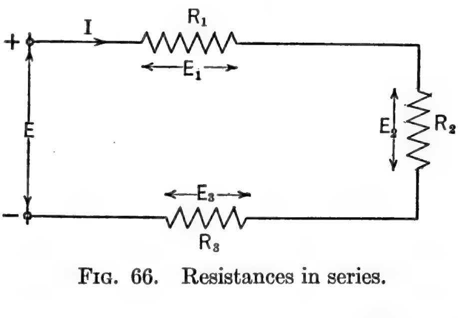


FIG. 66. Resistances in series.

cally equal to the sum of I_1 and I_2, and therefore the sum of all currents flowing toward the distributing point B is

$$I - (I_1 + I_2) = 0.$$

61. Resistances in Series. If a voltage E is applied across a circuit consisting of a number of resistances R_1, R_2 and R_3 connected in series (Fig. 66) the total resistance of the circuit will be equal to the sum of the resistances of the different parts.

The drop of voltage across the resistance R_1 is

$$E_1 = IR_1$$

where I is the current in the circuit.

Similarly

$$E_2 = IR_2,$$

and

$$E_3 = IR_3,$$

but

$$E = E_1 + E_2 + E_3$$
$$= I(R_1 + R_2 + R_3),$$

and the resistance of the whole circuit is

$$R = \frac{E}{I} = R_1 + R_2 + R_3. \quad . \quad . \quad . \quad . \quad (118)$$

62. Resistances in Parallel. Determine the resistance of a circuit consisting of a number of resistances R_1, R_2 and R_3 (Fig. 67) connected in parallel.

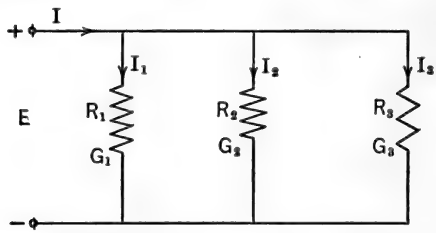


FIG. 67. Resistances in parallel.

If an e.m.f. E is applied to the circuit a current I will flow which will divide up into branch currents I_1, I_2 and I_3. By Ohm's law

$$I_1 = \frac{E}{R_1},$$

$$I_2 = \frac{E}{R_2},$$

$$I_3 = \frac{E}{R_3},$$

and

$$I = \frac{E}{R},$$

where R is the resistance of the whole circuit,

but

$$I = I_1 + I_2 + I_3$$

$$= E\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right),$$

therefore,

$$R = \frac{E}{E\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)}$$

$$= \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}. \quad \ldots \ldots \quad (119)$$

If the conductances G, G_1, G_2 and G_3 are used instead of the terms $\frac{1}{R}$, $\frac{1}{R_1}$, $\frac{1}{R_2}$ and $\frac{1}{R_3}$ the result can be written

$$R = \frac{1}{G} = \frac{1}{G_1 + G_2 + G_3},$$

or

$$G = G_1 + G_2 + G_3; \quad \ldots \ldots \quad (120)$$

that is, the conductance of a circuit consisting of a number of parallel branches is equal to the sum of the conductances of the branches.

63. The Potentiometer. The potentiometer shown in Fig. 68 gives a means of obtaining a variable voltage E_1 from a constant supply voltage E in a continuous current circuit.

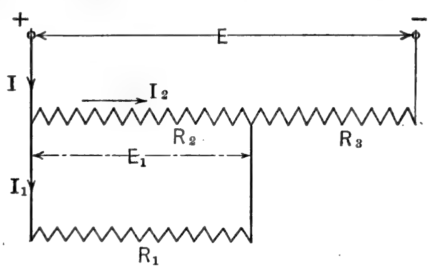


FIG. 68. Potentiometer.

Determine the voltage E_1 across the circuit AB and the current I_1 in the circuit in terms of the applied E_1 and the resistances, R_1, R_2 and R_3 as indicated in the figure.

Now

$$I = I_1 + I_2,$$

and

$$I_1R_1 = I_2R_2,$$

and

$$\begin{aligned} E &= IR_3 + I_2R_2 \\ &= (I_1 + I_2)\,R_3 + I_2R_2 \\ &= I_1R_3 + I_2\,(R_2 + R_3); \end{aligned}$$

but

$$I_2 = I_1\frac{R_1}{R_2},$$

therefore,

$$E = I_1R_3 + I_1\frac{R_1}{R_2}(R_2 + R_3),$$

and

$$I_1 = \frac{E}{R_3 + \frac{R_1}{R_2}(R_2 + R_3)},$$

and

$$E_1 = I_1R_1 = \frac{ER_1}{R_3 + \frac{R_1}{R_2}(R_2 + R_3)}.$$

The power lost in the potentiometer is

$$p_1 = I_2^2R_2 + I^2R_3;$$

the power output is

$$P = I_1^2R_1;$$

the efficiency of the potentiometer is

$$\begin{aligned} \eta &= \frac{\text{output}}{\text{input}} \times 100 \text{ per cent} = \frac{\text{output}}{\text{output} + \text{losses}} \times 100 \text{ per cent} \\ &= \frac{P}{P + p} \times 100 \text{ per cent} = \frac{I_1^2R_1}{I_1^2R_1 + I_2^2R_2 + I^2R_3} \times 100 \text{ per cent}; \end{aligned}$$

but $\quad I_2 = I_1\frac{R_1}{R_2} \quad$ and $\quad I = I_1 + I_2 = I_1\left(1 + \frac{R_1}{R_2}\right),$

and therefore

$$\begin{aligned} \eta &= \frac{I_1^2R_1}{I_1^2\left\{R_1 + \left(\frac{R_1}{R_2}\right)^2 R_2 + \left(\frac{R_1 + R_2}{R_2}\right)^2 R_3\right\}} \times 100 \text{ per cent} \\ &= \frac{R_1}{R_1 + \frac{R_1^2}{R_2} + \left(\frac{R_1 + R_2}{R_2}\right)^2 R_3} \times 100 \text{ per cent.} \end{aligned}$$

This method of varying the voltage supplied to a receiver circuit is very inefficient.

64. Inductance. When a current of electricity flows in a circuit, lines of magnetic induction interlink with the circuit; if the current remains constant the flux threading the circuit is constant, but when the current varies the flux varies proportionally and in doing so generates in the circuit an e.m.f. proportional to the rate of change of the interlinkages of flux and turns or to the product of the turns and the rate of change of the flux. This e.m.f. has a value

$$e = -n\frac{d\phi}{dt} \text{ c.g.s. units}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

where n is the number of turns in the circuit and $\frac{d\phi}{dt}$ is the rate of change of the flux interlinking with the turns. The negative sign is used because the e.m.f. produced opposes the change in the current and, therefore, the change in the flux.

The number of interlinkages of flux and turns for unit current in the circuit is called the inductance of the circuit and is represented by $\mathcal{L}$, thus,

$$\mathcal{L} = \frac{n\phi}{i} \text{ c.g.s. units}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

where i is the current in c.g.s. units,
ϕ is the flux produced by current i,
$n\phi$ is the number of interlinkages for current i.

Equation 122 may be written,

$$n\phi = \mathcal{L}i;$$

differentiating this gives

$$n\frac{d\phi}{dt} = \mathcal{L}\frac{di}{dt},$$

and therefore the e.m.f. generated in the circuit due to its inductance is

$$e = -n\frac{d\phi}{dt} = -\mathcal{L}\frac{di}{dt} \text{ c.g.s. units.} \quad . \quad . \quad . \quad . \quad (123)$$

It is equal to the product of the inductance of the circuit and the rate of change of the current. When e is expressed in volts and i in amperes the inductance is in henrys and is represented by L to distinguish it from the inductance $\mathcal{L}$ expressed in c.g.s. units. Equation 123 may then be written

$$e = -L\frac{di}{dt} \text{ volts.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

The henry is defined as the inductance of a circuit in which a counter e.m.f. of one volt is generated when the current is changing at the rate of one ampere per second.

The relation between the two units of inductance is found as follows:

$$\begin{aligned} e &= -\mathcal{L}\frac{di}{dt} \text{ c.g.s. units} \\ &= -\mathcal{L}\frac{di}{dt}10^{-8} \text{ volts, } \mathcal{L} \text{ and } i \text{ in c.g.s. units,} \\ &= -\mathcal{L}\frac{di}{dt}10^{-9} \text{ volts, } \mathcal{L} \text{ in c.g.s. units, } i \text{ in amperes,} \\ &= -L\frac{di}{dt} \text{ volts,} \end{aligned}$$

and, therefore,

$$1 \text{ henry} = 10^{-9} \text{ c.g.s. units of inductance.} \quad . \quad . \quad (125)$$

65. Inductance of Circuits Containing Iron. If a solenoid with n turns is wound on an iron ring of section A sq. cm. and mean length l cm., the flux produced by a current i c.g.s. units is

$$\phi = \frac{4\,\pi n i}{\dfrac{l}{A\mu}} \text{ lines,}$$

where μ is the permeability of the iron; the flux per unit current is

$$\phi_1 = \frac{\phi}{i} = \frac{4\,\pi n A \mu}{l} \text{ lines,}$$

and the inductance is

$$\mathcal{L} = n\phi_1 = \frac{4\,\pi n^2 A \mu}{l} \text{ c.g.s. units.}$$

It is proportional to the square of the number of turns as seen in the last example, but it also depends on the permeability of the iron and is therefore not a constant quantity. When the current in the solenoid is small, the iron is unsaturated, the permeability is high, the flux per unit current is large and therefore the inductance is large. When the current is large, the iron is saturated, the permeability is low, the flux per unit current is small and therefore the inductance is small.

The inductance of circuits containing iron or other magnetic materials is not constant, but decreases as the density of the induction increases, since the permeability decreases. (See Art. 49.)

66. Example. (a) Find the inductance of an endless solenoid in the form of a ring, Fig. 49.

n = number of turns in solenoid = 1000.
r = mean radius of the ring = 10 cm.
$l = 2\pi r$ = mean length of the flux path through the solenoid.
A = sectional area of the solenoid = 5 sq. cm.
i = current in solenoid in c.g.s. units.

The flux through the solenoid is

$$\phi = \frac{\text{m.m.f}}{\text{reluctance}} = \frac{4\pi ni}{l/A} \text{ lines,}$$

the flux per unit current is

$$\phi_1 = \frac{\phi}{i} = \frac{4\pi nA}{l},$$

the interlinkages of flux and turns per unit current is

$$\mathcal{L} = n\phi_1 = \frac{4\pi n^2 A}{l}$$

and is equal to the inductance of the circuit in c.g.s. units; thus, the inductance of a circuit is proportional to the square of the number of turns.

Substituting the values given above the inductance of the solenoid is

$$\mathcal{L} = \frac{4 \times 3.14 \times (1000)^2 \times 5}{2 \times 3.14 \times 10} = 10^6 \text{ c.g.s. units;}$$

and the inductance in practical units is

$$L = 10^6 \times 10^{-9} = 0.001 \text{ henry.}$$

(b) Find the energy stored in the magnetic field of the solenoid when the current has reached a value I = 10 c.g.s units.

The energy stored when any current i is flowing is equal to the work done in building up the current i against the counter e.m.f. of inductance $e = -\mathcal{L}\frac{di}{dt}$; it is therefore

$$W = \int ei\,dt = \int \mathcal{L}\frac{di}{dt} i\,dt = \mathcal{L}\int_0^I i\,di$$

$$= \mathcal{L}\frac{i^2}{2} \text{ ergs.}$$

When the current is $i = I = 10$ c.g.s. units, the energy stored is

$$W = \mathcal{L}\frac{I^2}{2} = 10^6 \times \frac{10^2}{2} = 5 \times 10^7 \text{ ergs}$$

$$= 5 \times 10^7 \times 10^{-7} = 5 \text{ watt-seconds.}$$

67. Mutual Inductance and Self-inductance. The mutual inductance of one circuit upon a second circuit is the number of interlinkages of the second circuit with the flux produced by unit current in the first circuit. It is equal to the mutual inductance of the second circuit upon the first and is represented by $\mathfrak{M}$.

A and *B*, in Fig. 69, are two electric circuits of n_1 and n_2 turns respectively interlinked with one magnetic circuit of reluctance $\mathfrak{R}$.

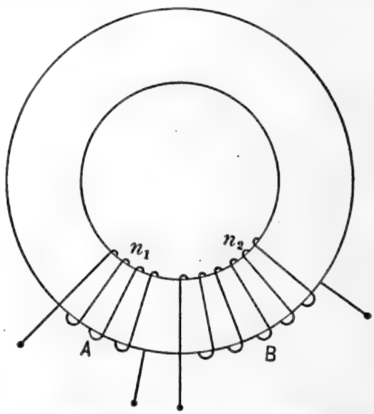


FIG. 69. Mutual inductance.

The flux produced by unit current in *A* is

$$\phi_1 = \frac{4\,\pi n_1}{\mathfrak{R}},$$

and the inductance of *A* is

$$\mathcal{L}_1 = n_1\phi_1 = \frac{4\,\pi n_1{}^2}{\mathfrak{R}} \text{ c.g.s. units;}$$

if all the flux produced by *A* passes through *B*, then, the mutual inductance of *A* upon *B* is by definition

$$\mathfrak{M} = n_2\phi_1 = \frac{4\,\pi n_1 n_2}{\mathfrak{R}} \text{ c.g.s. units. . . . .} \qquad (126)$$

Similarly the flux produced by unit current in B is

$$\phi_2 = \frac{4\pi n_2}{\mathfrak{R}}$$

and the inductance of B is

$$\mathcal{L}_2 = n_2\phi_2 = \frac{4\pi n_2^2}{\mathfrak{R}};$$

again if all the flux produced by B passes through A, the mutual inductance of B upon A is

$$\mathfrak{M} = n_1\phi_2 = \frac{4\pi n_1 n_2}{\mathfrak{R}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

and is equal to the mutual inductance of A upon B.

The leakage flux, that is, the flux which passes out between the two circuits and links with only one of them has been assumed to be very small. In this case

$$\mathcal{L}_1\mathcal{L}_2 = \frac{4\pi n_1^2}{\mathfrak{R}} \times \frac{4\pi n_2^2}{\mathfrak{R}} = \left(\frac{4\pi n_1 n_2}{\mathfrak{R}}\right)^2 = \mathfrak{M}^2. \quad . \quad . \quad . \quad (128)$$

If the leakage flux is not negligible, the flux ϕ_1 produced by unit current in A can be divided into two parts ϕ_{M_1} and ϕ_{s_1},

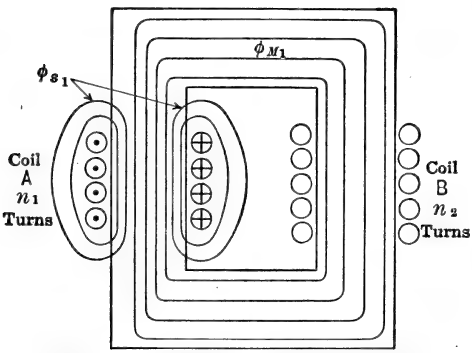


FIG. 70. Self and mutual inductance.

Fig. 70, of which ϕ_{M_1} interlinks with B and ϕ_{s_1} forms a local leakage circuit about A and does not interlink with B; ϕ_{s_1} is the self-inductive flux produced by unit current in A.

The self-inductance of a circuit is the number of interlinkages with the circuit of the flux produced by unit current in it, which does not interlink with any other circuit; it is the number of interlinkages of the circuit with the stray flux of the leakage flux produced by unit current in itself.

$$n_1\phi_1 = \mathcal{L}_1 = \text{inductance of } A,$$
$$n_2\phi_{M_1} = \mathfrak{M} = \text{mutual inductance of } A \text{ upon } B,$$
$$n_1\phi_{s_1} = \mathcal{L}_{s_1} = \text{self-inductance of } A,$$

but
$$\phi_1 = \phi_{M_1} + \phi_{s_1}, \text{ and therefore}$$

$$\frac{\mathcal{L}_1}{n_1} = \frac{\mathfrak{M}}{n_2} + \frac{\mathcal{L}_{s_1}}{n_1}, \quad \ldots \ldots \ldots \ldots \quad (129)$$

and, similarly,

$$\frac{\mathcal{L}_2}{n_2} = \frac{\mathfrak{M}}{n_1} + \frac{\mathcal{L}_{s_2}}{n_2}. \quad \ldots \ldots \ldots \ldots \quad (130)$$

When there is no mutual inductance the inductance of a circuit is the same as its self-inductance.

68. Self-Inductance of Continuous Current Circuits. The self-inductance of continuous current circuits is only apparent when the current is increasing or decreasing. The two most important cases are when the current is starting and when it is stopping.

Starting of Current. When a constant electromotive force E is impressed on a circuit of resistance R and inductance L the current does not immediately reach a steady value on account of the opposing e.m.f. due to inductance. If at time t after E is impressed the current is changing at the rate

$$\frac{di}{dt},$$

the e.m.f. of inductance is

$$e_b = -L\frac{di}{dt}.$$

By Lenz's law it opposes the impressed e.m.f. and is therefore negative.

The e.m.f. acting on the circuit is

$$E + e_b = E - L\frac{di}{dt},$$

and the current is

$$i = \frac{E - L\frac{di}{dt}}{R};$$

therefore
$$i - \frac{E}{R} = -\frac{L}{R}\frac{di}{dt},$$

or, transposing,
$$\frac{di}{i - \frac{E}{R}} = -\frac{R\,dt}{L};$$

the integral of this is

$$\log_\epsilon\left(i - \frac{E}{R}\right) = -\frac{Rt}{L} + \log_\epsilon C,$$

where $\log_\epsilon C$ is the constant of integration. This reduces to

$$i - \frac{E}{R} = C\epsilon^{-\frac{Rt}{L}},$$

or

$$i = \frac{E}{R} + C\epsilon^{-\frac{Rt}{L}},$$

when $t = 0$, $i = 0$, and $C = -\frac{E}{R}$.

Substituting this value the current is

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right). \quad \quad (131)$$

When $t = \infty$ and $i = \frac{E}{R}$,

and the current has reached its steady value, which is independent of the inductance of the circuit; calling this value I and substituting

$$i = I\left(1 - \epsilon^{-\frac{Rt}{L}}\right). \quad \quad (132)$$

The expression $\frac{R}{L}$ is called the time constant of the circuit.

Stopping of Current. If the impressed electromotive force E is removed from a circuit of resistance R and inductance L, when it is carrying a current $I = \frac{E}{R}$, and the circuit is closed through a resistance R_1, the current will not at once fall to zero.

The e.m.f. of inductance at time t after the circuit is closed is

$$e_b = -L\frac{di}{dt},$$

and it opposes the decrease of current.

The current in the circuit at this instant is

$$i = -\frac{L\frac{di}{dt}}{R + R_1},$$

or, transposing,

$$\frac{di}{i} = -\frac{(R + R_1)}{L}dt,$$

the integral of this is

$$\log_\epsilon i = -\frac{R + R_1}{L}t + \log C,$$

where log C is the constant of integration. This reduces to

$$i = C\epsilon^{-\frac{R+R_1}{L}t},$$

when $t = 0$, $i = I$, and $C = I$.

Substituting this value, the current is

$$i = I\epsilon^{-\frac{R+R_1}{L}t}; \quad \ldots \ldots \ldots \quad (133)$$

or

$$i = \frac{E}{R}\epsilon^{-\frac{R+R_1}{L}t}. \quad \ldots \ldots \ldots \quad (133a)$$

The e.m.f. generated in the coil, at the time t, is

$$e_b = i(R + R_1) = E\frac{R + R_1}{R}\epsilon^{-\frac{R+R_1}{L}t}. \quad \ldots \quad (134)$$

When $t = 0$, the generated e.m.f. is

$$e_b = E\frac{R + R_1}{R}; \quad \ldots \ldots \ldots \quad (134a)$$

therefore the e.m.f. generated in the circuit is greater than the impressed e.m.f. in the ratio $\frac{R + R_1}{R}$.

If, when the impressed e.m.f. is removed, the circuit is short-circuited, $R_1 = 0$ and the generated e.m.f. is

$$e_b = E,$$

that is, the e.m.f. does not rise.

But if the circuit is broken $R_1 = \infty$ and $e_b = \infty$, and dangerous e.m.f.'s are generated in the circuit. A large value of R_1 causes the current to decrease rapidly, but it also causes a high e.m.f. to be generated.

The energy supplied to the circuit while the current is starting is

$$W = \int_0^\infty Ei\,dt,$$

but

$$E = Ri + L\frac{di}{dt};$$

therefore,

$$W = \int_0^\infty \left(Ri + L\frac{di}{dt}\right) i\,dt = \int_0^\infty Ri^2\,dt + \int_0^I Li\,di;$$

of this $\int_0^\infty Ri^2\,dt$ is the energy transformed into heat in the electric circuit, and $\int_0^I Li\,di$ is the energy stored in the magnetic field.

The energy in the magnetic field is

$$W_M = \int_0^I Li\,di = L\frac{I^2}{2}$$

and is independent of the resistance of the circuit.

This magnetic energy is returned to the electric circuit while the current is stopping and prevents it from immediately falling to zero. It is transformed into heat in the circuit and is

$$W = \int_0 i^2 (R + R_1)\,dt,$$

but

$$i = Ie^{-\frac{R+R_1}{L}t}.$$

therefore,

$$W = I^2 (R + R_1) \int_0^\infty e^{-2\frac{(R+R_1)}{L}t}\,dt;$$

the integral of this is

$$W = I^2 (R + R_1) - \frac{L}{2(R + R_1)}\left[e^{-2\left(\frac{R+R_1}{L}\right)t}\right]_0^\infty,$$

and

$$W = L\frac{I^2}{2} \text{ as before.}$$

If the circuit is broken this amount of energy must be discharged and produces a spark.

69. Example. The shunt field winding of a 6-pole, 125-volt, direct-current generator has 500 turns per pole and carries 12.5 amperes; the resistance of the field winding is 7.5 ohms and the resistance of the field rheostat is $\frac{125}{12.5} - 7.5 = 2.5$ ohms; and the flux per pole is 6.3×10^6 lines; determine:

(a) the inductance of the field winding;
(b) the time taken by the current to reach $\frac{1}{10}$ full value and $\frac{1}{2}$ full value, after the field switch is closed;
(c) the initial rise of voltage, if the field circuit carrying a current of 12.5 amperes is closed through a resistance of 25 ohms;
(d) the energy stored in the magnetic field when the current is 12.5 amperes.

(a) The interlinkages of flux and turns with a current of 12.5 amperes or 1.25 c.g.s. units is

$$6 \times 500 \times 6.3 \times 10^6;$$

therefore the inductance is

$$\mathcal{L} = \frac{6 \times 500 \times 6.3 \times 10^6}{1.25} = 15.1 \times 10^9 \text{ c.g.s. units,}$$

or

$$L = 15.1 \text{ henrys.}$$

(b) At time t after the field switch is closed the current is, by equation 132,

$$i = I\left(1 - \epsilon^{-\frac{Rt}{L}}\right),$$

where I is the final value of the current = 12.5 amperes and R is the resistance of the field circuit including the rheostat = 10 ohms; therefore,

$$i = 12.5\left(1 - \epsilon^{-\frac{10\,t}{15.1}}\right).$$

Let time for the current to reach $\frac{1}{10}$ of full value or 1.25 amperes be t_1 sec., then

$$1.25 = 12.5\,(1 - \epsilon^{-0.66\,t_1})$$

and

$$t_1 = 0.16 \text{ sec.}$$

Let time for the current to reach $\frac{1}{2}$ value or 6.25 amperes be t_2 sec., then

$$6.25 = 12.5\,(1 - \epsilon^{-0.66\,t_2})$$

and

$$t_2 = 1.05 \text{ sec.}$$

(c) If the field circuit carrying a current of 12.5 amperes is closed through a resistance of 25 ohms, the initial value of the back voltage due to inductance is, by equation 134*a*,

$$e_b = 125\,\frac{10 + 25}{10} = 437.5 \text{ volts.}$$

(d) When the current is 12.5 amperes the energy stored in the magnetic field is

$$W = L\frac{I^2}{2} = 15.1 \times \frac{(12.5)^2}{2} = 1460 \text{ watt-seconds.}$$

70. Example. Determine the inductance of the circuit in Fig. 71 consisting of two parallel wires A and B carrying equal currents but in opposite directions.

R = radius of each wire in centimeters.
D = distance between centres in centimeters.

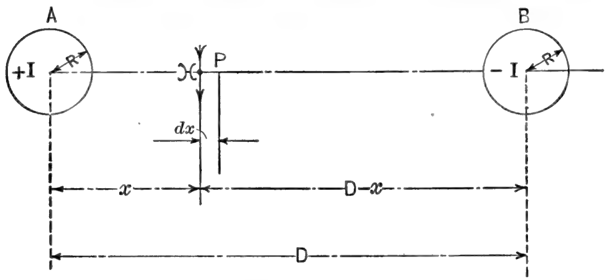


FIG. 71. Inductance of parallel lines.

When a current I c.g.s. units flows in the wires the field intensity at the point P distant x cm. from A and $D-x$ cm. from B is

$$\mathcal{H} = \frac{2I}{x} + \frac{2I}{D-x} \text{ dynes, by equation 90,}$$

and the flux density is

$$\mathcal{B} = \mathcal{H} = \frac{2I}{x} + \frac{2I}{D-x} \text{ lines per square centimeter;}$$

the flux in a section of width dx and length 1 cm. is

$$d\Phi = \mathcal{B}\, dx \text{ lines;}$$

and the total flux between the wires per centimeter length is

$$\Phi = \int_R^{D-R} \mathcal{B}\, dx = \int_R^{D-R} \left(\frac{2I}{x} + \frac{2I}{D-x}\right) dx$$

$$= 2I\Big[\log x - \log (D-x)\Big]_R^{D-R}$$

$$= 2\left[\log\frac{D-R}{R} - \log\frac{R}{D-R}\right]$$

$$= 2I\left(2I \log\frac{D-R}{R}\right),$$

but only half of this flux surrounds each wire and therefore the inductance of each wire per centimeter length is

$$L_1 = \frac{\Phi/2}{I} = 2 \log \frac{D - R}{R} \text{ c.g.s. units.} \quad . \quad . \quad . \quad (135)$$

The flux inside the conductor has been neglected, but its inductive effect can be calculated very easily if it is assumed that the current is distributed uniformly over the section of the wire. In Fig. 72 the section of wire A is shown enlarged.

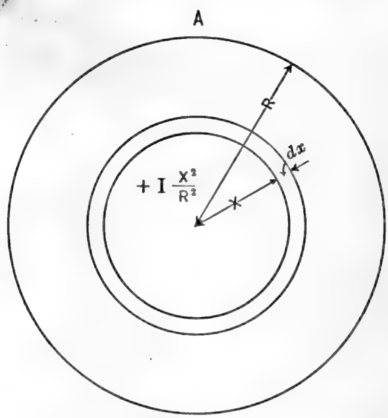


FIG. 72. Flux inside a conductor.

The flux density at distance x cm. from the centre of A is

$$\mathfrak{B} = \frac{2\,I'}{x},$$

where I' is the current inside the radius x cm. and its value is

$$I' = I \frac{x^2}{R^2} \text{ c.g.s. units.}$$

The flux in the ring of radius x, width dx and length 1 cm. is

$$d\phi' = \frac{2\,I'}{x}\,dx = \frac{2\,Ix}{R^2}\,dx \text{ lines;}$$

this flux surrounds only the current I' and is equivalent to a smaller flux surrounding the current I of value

$$d\phi = d\phi' \frac{x^2}{R^2} = \frac{2\,Ix^3}{R^4}\,dx,$$

and the flux equivalent to the flux inside the whole section is

$$\phi_2 = \int_0^R d\phi = \int_0^R \frac{2I}{R^4} x^3\,dx$$
$$= \frac{2I}{R^4}\left[\frac{x^4}{4}\right]_0^R = \frac{I}{2} \text{ lines,}$$

and the inductance per centimeter due to the flux inside the conductor is

$$\mathcal{L}_2 = \frac{\phi_2}{I} = \frac{1}{2} \text{ c.g.s. units.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

The inductance of each wire per centimeter is therefore

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = 2 \log \frac{D-R}{R} + \frac{1}{2} \text{ c.g.s. units,} \quad . \quad . \quad . \quad (137)$$

and the inductance per mile of wire is

$$L = 2.54 \times 12 \times 5280\left(2 \times 2.303 \log_{10} \frac{D-R}{R} + 0.5\right)10^{-9} \text{ henrys}$$
$$= \left(0.74 \log_{10} \frac{D-R}{R} + 0.0805\right)10^{-3} \text{ henrys.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

71. The Sine Wave of Electromotive Force and Current. If the coil *abcd* in Fig. 73 rotates with a constant angular velocity in a uniform magnetic field between the poles N and S, an alter-

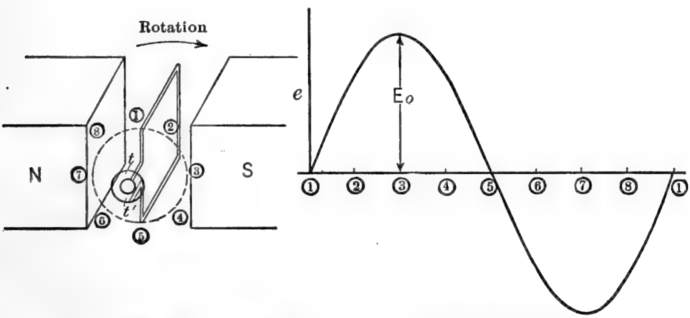


FIG. 73. Generation of an alternating e.m.f.

nating e.m.f. will be generated between the terminals t and t'. Referring to the figure it can be seen that in position (1) the e.m.f. generated in the coil is zero, since the conductors forming it are moving parallel to the lines of magnetic flux and therefore are not cutting them; in position (2) it is positive and increasing; in

(3) it is positive and has reached its maximum value since the conductors are cutting perpendicularly across the flux; in (4) it is positive and decreasing; in (5) it is zero; in (6) it is negative and increasing; in (7) negative and maximum; in (8) negative and decreasing and in (1) is zero again, having gone through one complete cycle. This cycle is represented in Fig. 73.

If Φ = the maximum flux inclosed by the coil, n = the number of turns on the coil and ω = the angular velocity in radians per second, then at time t sec. after the position of maximum inclosure the coil has moved through angle $\theta = \omega t$ radians, and the flux inclosed is

$$\phi = \Phi \cos \omega t;$$

the e.m.f. generated in the coil at this instant is

$$\begin{aligned} e &= - n \frac{d}{dt} (\Phi \cos \omega t) \\ &= \omega n \Phi \sin \omega t \text{ absolute units} \\ &= \omega n \Phi \, 10^{-8} \sin \omega t \text{ volts.} \quad \quad (139) \end{aligned}$$

When $\theta = \frac{\pi}{2}$, the e.m.f. has its maximum value

$$E_0 = \omega n \Phi \, 10^{-8} \text{ volts,}$$

and therefore

$$e = E_0 \sin \omega t; \quad \quad (140)$$

and the e.m.f. generated in the coil varies as a sine wave.

The number of cycles through which the e.m.f. passes in one second is called its frequency and since one cycle represents 360 electrical degrees, the frequency may be expressed as

$$f = \frac{\omega}{2\pi} \text{ cycles,} \quad \quad (141)$$

and therefore

$$\omega = 2\pi f.$$

Substituting this value of ω in equation 139 gives

$$\begin{aligned} e &= 2\pi f n \Phi \, 10^{-8} \sin 2\pi f t \\ &= E_0 \sin \theta, \end{aligned}$$

where $\theta = 2\pi f t$ is the angle turned through in time t sec. after the position of zero e.m.f.; the maximum value of the e.m.f. is

$$E_0 = 2\pi f n \Phi \, 10^{-8} \text{ volts.} \quad \quad (142)$$

In a two-pole alternating-current generator one cycle corresponds to one revolution of the coil and

$$f = \text{rev. per sec.},$$

and one electrical space degree is equal to one mechanical space degree. With a p-pole alternator the e.m.f. passes through one cycle for each pair of poles and the frequency is

$$f = \frac{p}{2} \text{rev. per sec.},$$

and one electrical space degree is less than one mechanical space degree in the ratio $\frac{2}{p}$.

If a non-inductive resistance of R ohms is connected across the

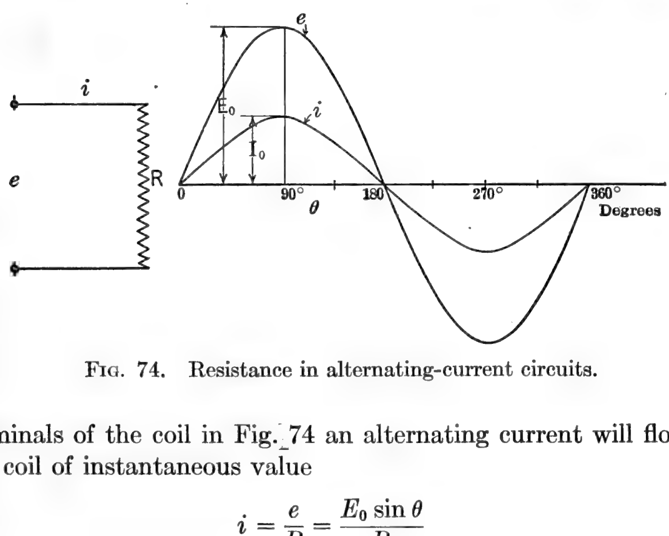


FIG. 74. Resistance in alternating-current circuits.

terminals of the coil in Fig. 74 an alternating current will flow in the coil of instantaneous value

$$i = \frac{e}{R} = \frac{E_0 \sin \theta}{R}$$

$$= I_0 \sin \theta, \quad \ldots \ldots \ldots \quad (143)$$

where $I_0 = \frac{E_0}{R}$ is the maximum value of the current. The current and voltage waves pass through their zero and maximum values together and are therefore in phase. They are represented by the two curves in Fig. 74.

72. The Average Value of a Sine Wave. The average value of the ordinate of a sine wave $i = I_0 \sin \theta$ can be found by in-

tegrating over one-half wave; it is

$$
\begin{aligned}
I_{avg} &= \frac{1}{\pi}\int_0^{\pi} I_0 \sin\theta \, d\theta \\
&= \frac{I_0}{\pi}\Big[-\cos\theta\Big]_0^{\pi} \\
&= \frac{I_0}{\pi}[1+1] = \frac{2}{\pi} I_0 = 0.637\, I_0, \qquad (144)
\end{aligned}
$$

that is, the average value of the ordinate of a sine curve is $\frac{2}{\pi}$ times the maximum ordinate.

73. The Effective Value of a Sine Wave. The effective value of an alternating current is the value of continuous current which would have the same heating effect.

When an alternating current $i = I_0 \sin\theta$ flows through a resistance R, energy is transformed into heat at the instantaneous rate i^2R watts; the average rate of transformation of energy is

$$
\begin{aligned}
\frac{1}{\pi}\int_0^{\pi} i^2R \, d\theta &= \frac{1}{\pi}\int_0^{\pi} I_0^2R \sin^2\theta \, d\theta \\
&= \frac{I_0^2R}{2\pi}\int_0^{\pi}(1-\cos 2\theta)\, d\theta \\
&= \frac{I_0^2R}{2\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} \\
&= \frac{I_0^2R}{2\pi}[\pi] \\
&= \frac{I_0^2R}{2} = \left(\frac{I_0}{\sqrt{2}}\right)^2 R; \qquad (145)
\end{aligned}
$$

thus the alternating current $i = I_0 \sin\theta$ has the same average heating effect or consumes the same average power as a continuous current $\frac{I_0}{\sqrt{2}}$; the value $I = \frac{I_0}{\sqrt{2}}$ is therefore called the effective value of the alternating current $i = I_0 \sin\theta$.

The effective value of the alternating e.m.f. $e = E_0 \sin\theta$ is

$$
E = \frac{E_0}{\sqrt{2}}. \qquad (146)
$$

The effective value of any alternating quantity is the square root of the mean of the squared instantaneous values taken over

one complete cycle; it is equal to the maximum value divided by $\sqrt{2}$ only in the case of sine waves.

Alternating-current voltmeters and ammeters indicate the effective values of voltage and current regardless of the wave form.

74. Inductance in Alternating-current Circuits. When a current flows in a conductor a magnetic field is produced in the space surrounding it; as long as the current remains constant this field does not react on the electric circuit, but when the current varies the flux linking with the circuit also varies and induces in the conductor an e.m.f. opposing the change in the current and consequently the change in the flux. This action is due to the inertia of the magnetic field and is analogous to the action of the flywheel in mechanics. The inertia of the flywheel opposes any change in speed just as the inertia of the magnetic field opposes any change in current. Energy is stored in the flywheel as the speed increases and given back as it decreases and the only loss of energy is that due to friction. Similarly energy is stored in the magnetic field as the current increases and is returned to the electric circuit as the current decreases, and the only loss of energy is that due to hysteresis and eddy currents in the iron parts of the magnetic circuit.

The energy stored in the flywheel is

$$W = I\frac{\omega^2}{2}, \quad \dots\dots\dots \quad (147)$$

where I is its moment of inertia and ω is its angular velocity.

The energy stored in the magnetic field is

$$W = \frac{Li^2}{2}\text{ watt-seconds,}$$

where L is the inductance of the circuit in henrys and i is the current in amperes.

The inductance of the coil opposes the change in current by generating a back e.m.f.

$$e_b = -L\frac{di}{dt}.$$

If an alternating current $i = I_0 \sin \theta = I_0 \sin 2\pi ft$ is flowing through a circuit of inductance L henrys and negligible resistance an e.m.f. of inductance will be set up

$$\begin{aligned} e_b &= - L \frac{di}{dt} \\ &= - L \frac{d}{dt}(I_0 \sin 2\pi ft) \\ &= - 2\pi fLI_0 \cos 2\pi ft \\ &= 2\pi fLI_0 \sin (2\pi ft - 90); \quad \ldots \ldots \quad (148) \end{aligned}$$

it is a sine wave of e.m.f. lagging behind the current wave by 90 degrees. (Fig. 75.)

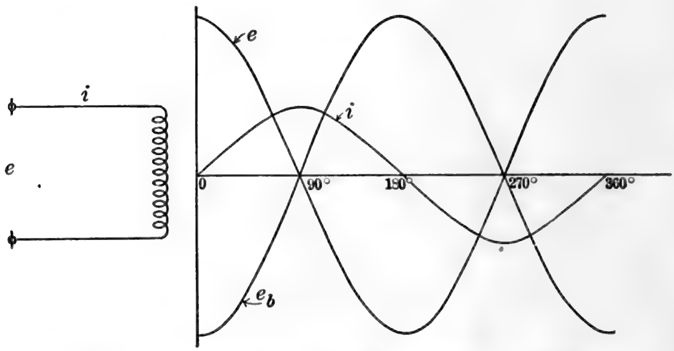


FIG. 75. Inductance in alternating current circuits.

In order to drive the current through the circuit an e.m.f. must be applied to the terminals, which at every instant will be equal and opposite to the back e.m.f. e_b; its instantaneous value is

$$e = - e_b = 2\pi fLI_0 \sin (2\pi ft + 90), \quad \ldots \quad (149)$$

a sine wave of e.m.f. leading the current wave by 90 degrees.

This e.m.f. is consumed by the inductance of the circuit. Its maximum value is

$$E_0 = 2\pi fLI_0, \quad \ldots \ldots \quad (150)$$

where the term $2\pi fL$ is called the inductive reactance of the circuit and is denoted by X. It is of the nature of resistance and is expressed in ohms.

Thus, in a circuit of reactance and negligible resistance the impressed e.m.f. is 90 degrees ahead of the current, or the current lags 90 degrees behind the impressed e.m.f.

75. Resistance and Reactance in Series. If a circuit contains a resistance R and a reactance X in series and carries an alternating current $i = I_0 \sin 2\pi ft$, determine the value of the impressed e.m.f. and its phase relation with the current. (Fig. 76.)

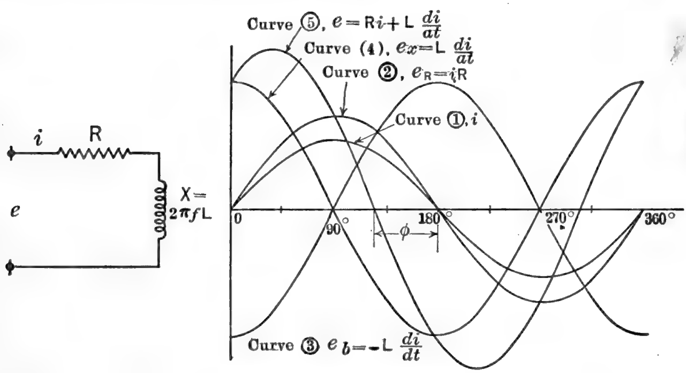


FIG. 76. Resistance and reactance in series.

To drive a current $i = I_0 \sin 2\pi ft$ (curve 1) through a resistance R an e.m.f. is required equal to

$$e_R = iR = I_0 R \sin 2\pi ft \text{ (curve 2)},$$

a sine wave in phase with the current with a maximum value $I_0 R$.

The inductance of the circuit sets up a back e.m.f.,

$$e_b = -L\frac{di}{dt} = 2\pi fLI_0 \sin(2\pi ft - 90) \text{ (curve 3)},$$

a sine wave lagging 90 degrees behind the current with a maximum value $2\pi fLI_0 = I_0 X$.

To overcome this back e.m.f. due to inductance an e.m.f. must be impressed on the circuit equal and opposite to e_b.

$$\begin{aligned} e_X = -e_b &= L\frac{di}{dt} \\ &= 2\pi fL\, I_0 \sin(2\pi ft + 90) \\ &= I_0 X \sin(2\pi ft + 90) \text{ (curve 4)}. \end{aligned}$$

This is a sine wave 90 degrees ahead of the current wave, with a maximum value $I_0 x$ and is the e.m.f. consumed by the reactance in the circuit.

The instantaneous value of the impressed e.m.f. (curve 5) is the sum of the instantaneous values of e_R and e_x, thus

$$e = e_R + e_X = Ri + L\frac{di}{dt} \quad \dots\dots\dots\dots \quad (151)$$

$$= I_0 R \sin 2\pi ft + I_0 X \sin (2\pi ft + 90)$$

$$= I_0 R \sin 2\pi ft + I_0 X \cos 2\pi ft$$

$$= I_0 (R \sin \theta + X \cos \theta)$$

$$= I_0 \sqrt{R^2+X^2}\left(\frac{R}{\sqrt{R^2+X^2}} \sin \theta + \frac{X}{\sqrt{R^2+X^2}} \cos \theta\right)$$

$$= I_0 Z \sin (\theta + \phi), \quad \dots\dots\dots\dots \quad (152)$$

where $Z = \sqrt{R^2+X^2}$ is called the impedance of the circuit and is expressed in ohms and ϕ is the angle of lead of the impressed e.m.f. relative to the current.

$$\sin \phi = \frac{X}{\sqrt{R^2+X^2}} \text{ and } \cos \phi = \frac{R}{\sqrt{R^2+X^2}}.$$

The impressed e.m.f. is therefore a sine wave leading the current by an angle ϕ. Its maximum value is

$$E_0 = I_0 \sqrt{R^2+X^2} = I_0 Z, \quad \dots\dots \quad (153)$$

and its effective value is

$$E = \frac{E_0}{\sqrt{2}} = \frac{I_0 Z}{\sqrt{2}} = IZ,$$

where I is the effective value of the current.

The component of E in phase with the current is

$$E_1 = IR = \frac{E}{Z} R = E \cos \phi;$$

and the component of E 90 degrees ahead or in quadrature ahead of the current is

$$E_2 = IX = \frac{E}{Z} X = E \sin \phi.$$

76. Capacity in Alternating-current Circuits. The charge on a condenser or the quantity of electricity stored in it is proportional to the difference of potential between its terminals; thus,

$$q = Ce,$$

where q is the charge,

e is the difference of potential between the terminals,

and C is the capacity of the condenser.

A condenser has a capacity of one farad when one coulomb of electricity stored in it produces a difference of potential of one volt between its terminals.

If an alternating e.m.f. $e = E_0 \sin 2\pi ft$ is impressed on the terminals of a condenser of capacity C farads, a current i flows. At any time t the charge on the condenser is

$$q = Ce,$$

but the charge is the amount of electricity which has flowed into the condenser and its value is

$$q = \int_0^t i\,dt,$$

and therefore

$$\int i\,dt = Ce.$$

Differentiating with respect to t gives

$$i = C\frac{de}{dt},$$

and substituting the value of e,

$$\begin{aligned} i &= C\frac{d}{dt}(E_0 \sin 2\pi ft) \\ &= 2\pi fCE_0 \cos 2\pi ft \\ &= 2\pi fCE_0 \sin (2\pi ft + 90), \quad \ldots \quad (154) \end{aligned}$$

and thus the current flowing into the condenser is a sine wave leading the impressed e.m.f. by 90 degrees. Its maximum value is

$$\begin{aligned} I_0 &= 2\pi fCE_0 \\ &= \frac{E_0}{X_c}, \quad \ldots \quad (155) \end{aligned}$$

where $X_c = \frac{1}{2\,\pi f C}$ is called the condensive reactance of the circuit and is expressed in ohms.

In Fig. 77 curve 1 represents the impressed e.m.f. $e = E_0$

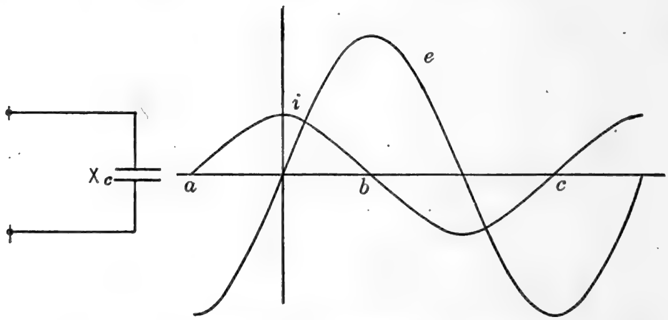


FIG. 77. Capacity in alternating current circuits.

$\sin 2\,\pi ft$ and curve 2 the current $i = I_0 \sin (2\,\pi ft + 90)$. Between the points a and b the current is positive and the e.m.f. is increasing; from b to c the current is negative and the e.m.f. is decreasing.

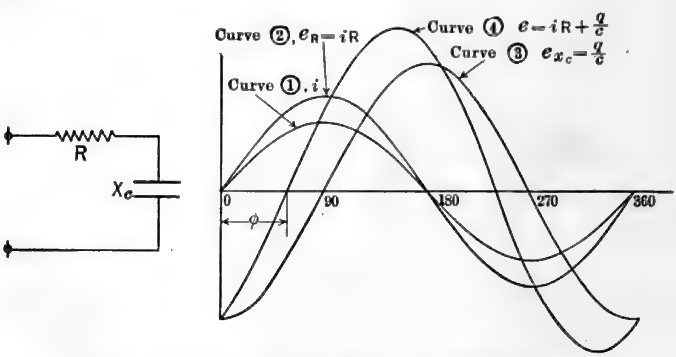


FIG. 78. Resistance and condensive reactance in series.

77. Resistance and Condensive Reactance in Series. If an alternating current $i = I_0 \sin 2\,\pi ft$ flows in a circuit consisting of a resistance R in series with a condenser of capacity C farads and reactance of $X_c = \frac{1}{2\,\pi f C}$ ohms, determine the magnitude of

the impressed e.m.f. and its phase relation with the current. (Fig. 78.)

The impressed e.m.f. consists of two components; one is required to drive the current i (curve 1) through the resistance of the circuit; it is $e_R = iR$ (curve 2), in phase with i; the other component is the e.m.f. required at the terminals of the condenser or the e.m.f. consumed by the condensive reactance of the circuit; it is $e_{X_c} = \frac{q}{C}$ and was shown to lag 90 degrees behind the current. Art. 76 (curve 3).

The total impressed e.m.f. is (curve 4)

$$e = e_R \dotplus \overline{e_{X_c}} \qquad (156)$$

$$= iR + \frac{q}{C}$$

$$= iR + \frac{\int i\,dt}{C}$$

$$= RI_0 \sin 2\pi ft + \frac{1}{C}\int I_0 \sin (2\pi ft)\, dt$$

$$= RI_0 \sin 2\pi ft - \frac{I_0}{2\pi fC} \cos 2\pi ft$$

$$= I_0 (R \sin 2\pi ft - X_c \cos 2\pi ft)$$

$$= I_0 \sqrt{R^2+X_c^2}\left(\frac{R}{\sqrt{R^2+X_c^2}} \sin 2\pi ft - \frac{X_c}{\sqrt{R^2+X_c^2}} \cos 2\pi ft\right)$$

$$= I_0 Z \sin (2\pi ft - \phi), \qquad (157)$$

where $Z = \sqrt{R^2+X_c^2}$ is the impedance of the circuit in ohms, and ϕ is the angle of lag of the impressed e.m.f. behind the current; $\sin \phi = \frac{X_c}{\sqrt{R^2+X_c^2}}$, $\cos \phi = \frac{R}{\sqrt{R^2+X_c^2}}$.

78. Resistance, Inductance and Capacity in Series. If a current $i = I_0 \sin 2\pi ft$ flows in a circuit consisting of a resistance R, inductance L and capacity C connected in series, the impressed e.m.f. is

$$e = e_R \dotplus e_X \dotplus e_{X_c}, \qquad (158)$$

where $e_R = iR$ is the e.m.f. consumed by the resistance of the circuit,

$e_X = L\frac{di}{dt}$ is the e.m.f. consumed by the inductance of the circuit,

$e_{X_c} = \frac{q}{C}$ is the e.m.f. consumed by the capacity of the circuit.

Assuming that the impressed e.m.f. leads the current by angle ϕ, its equation is

$$e = E_0 \sin(2\pi ft + \phi), \qquad (159)$$

and

$$E_0 \sin(2\pi ft + \phi) = Ri + L\frac{di}{dt} + \frac{q}{C}$$

$$= Ri + L\frac{di}{dt} + \frac{\int i\,dt}{C}. \qquad (160)$$

Differentiating with respect to t,

$$2\pi fE_0 \cos(2\pi ft + \phi) = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C}, \qquad (161)$$

but

$$i = I_0 \sin 2\pi ft, \quad \frac{di}{dt} = 2\pi fI_0 \cos 2\pi ft, \quad \frac{d^2i}{dt^2} = -(2\pi f)^2 I_0 \sin 2\pi ft$$

when $t = 0, \quad i = 0, \quad \frac{di}{dt} = 2\pi fI_0, \quad \frac{d^2i}{dt^2} = 0,$

when

$$t = \frac{90 - \phi}{2\pi f}, \quad i = I_0 \cos\phi, \quad \frac{di}{dt} = 2\pi fI_0 \sin\phi, \quad \frac{d^2i}{dt^2} = -(2\pi f)^2 I_0 \cos\phi.$$

Substituting these values in equation 161, when $t = 0$,

$$2\pi fE_0 \cos\phi = 2\pi fRI_0,$$

and

$$\frac{E_0}{I_0} = \frac{R}{\cos\phi}; \qquad (162)$$

when $t = \frac{90 - \phi}{2\pi f}$,

$$0 = 2\pi fRI_0 \sin\phi - (2\pi f)^2 I_0 \cos\phi + \frac{I_0}{C}\cos\phi,$$

and

$$R \sin \phi = 2 \pi f L \cos \phi - \frac{1}{2 \pi f C} \cos \phi,$$

therefore,

$$\tan \phi = \frac{2 \pi f L - \dfrac{1}{2 \pi f C}}{R}$$

$$= \frac{X - X_c}{R}, \quad \ldots \ldots \ldots \quad (163)$$

$$\sin \phi = \frac{X - X_c}{\sqrt{R^2 + (X - X_c)^2}}, \quad \ldots \ldots \ldots \quad (164)$$

$$\cos \phi = \frac{R}{\sqrt{R^2 + (X - X_c)^2}}, \quad \ldots \ldots \ldots \quad (165)$$

and from equation 162

$$\frac{E_0}{I_0} = \frac{R}{\cos \phi} = \sqrt{R^2 + (X - X_c)^2} = Z, \quad \ldots \quad (166)$$

where

$$Z = \sqrt{R^2 + (X - X_c)^2} = \sqrt{R^2 + \left(2 \pi f L - \frac{1}{2 \pi f C}\right)^2}$$

is the impedance of the circuit.

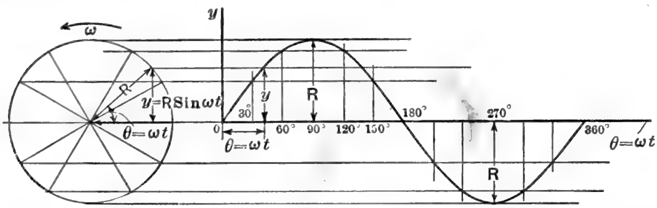


FIG. 79. Sine wave.

The impressed e.m.f. is therefore a sine wave of maximum value $E_0 = I_0 Z = I_0 \sqrt{R^2 + (X - X_c)^2}$, and leads the current wave by an angle $\phi = \tan^{-1} \dfrac{X - X_c}{R}$.

If $X_c > X$, ϕ is negative and the impressed e.m.f. lags behind the current.

If $X_c = X$, $\phi = 0$ and the e.m.f. is in phase with the current, and the impedance of the circuit is $Z = R$.

79. Vector Representation of Harmonic Quantities. A sine wave may be obtained from a circular locus as shown in Fig. 79.

The radius of the circle is the maximum value of the sine function and is called its amplitude. If the radius vector is rotated at uniform angular velocity ω in the counter-clockwise direction, its vertical projection at any time t is

$$y = R \sin \omega t = R \sin \theta,$$

where time is measured from the horizontal. Plotting the values of y on a base of angle θ for a complete revolution of the radius vector gives the cycle shown.

To represent the sine function

$$e = E_0 \sin \theta,$$

a circle of radius E_0 is taken and the vertical projections of the revolving vector are plotted on base of angle θ. (Fig. 80 (*a*).)

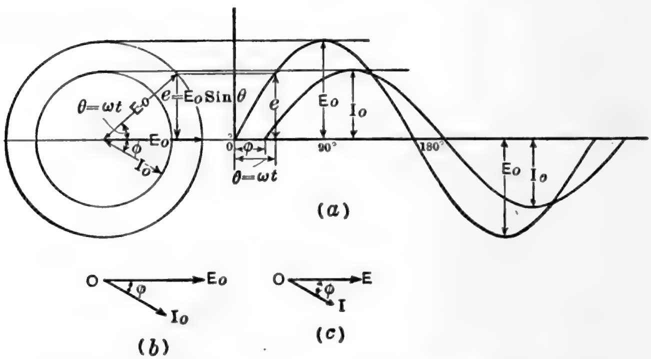


FIG. 80. Vector representation of harmonic functions.

To represent a sine function

$$i = I_0 \sin (\theta - \phi),$$

a circle of radius I_0 is taken, but since i does not pass through zero until angle $\theta = \phi$, the sine wave is displaced to the right by angle ϕ as shown and the e.m.f. wave $e = E_0 \sin \theta$ leads the current wave $i = I_0 \sin (\theta - \phi)$ by angle ϕ. Instead of using complete circles to represent sine waves, their radii E_0 and I_0 can be used as in Fig. 80 (*b*) or since alternating quantities are represented by their effective values instead of their maximum values the two vectors OE and OI, Fig. 80 (*c*), are used to represent the two sine waves in Fig. 80 (*a*). $OE = E = \dfrac{E_0}{\sqrt{2}}$ is the effective value

of the e.m.f. $e = E_0 \sin \theta$, and $OI = I = \frac{I_0}{\sqrt{2}}$ is the effective value of the current $i = I_0 \sin(\theta - \phi)$. The vector OI is behind the vector OE by angle ϕ and so indicates the relative phase of the two quantities. The counter-clockwise direction is taken as the direction of advance in phase since it was adopted at the International Electrotechnical Congress at Turin and is now standard.

80. Power and Power Factor. In continuous current circuits the power consumed is the product of the impressed e.m.f. and the current, and is

$$P = EI \text{ watts.} \qquad (167)$$

In alternating-current circuits the power varies with time; its instantaneous value is

$$p = ei \text{ watts,} \qquad (168)$$

where e and i are the instantaneous values of the e.m.f. and current in the circuit.

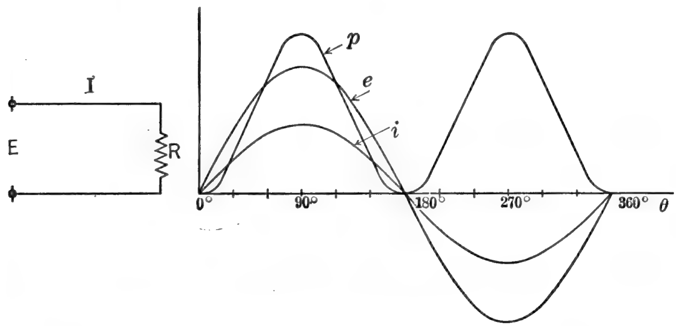


FIG. 81. Power in a non-inductive circuit.

Two cases will be considered, first, when the current and e.m.f. are in phase and, second, when the current lags behind the e.m.f.

Case I. If an e.m.f. $e = E_0 \sin \theta$ is impressed on the terminals of a circuit of resistance R a current $i = I_0 \sin \theta$ will flow in phase with the e.m.f.

The instantaneous power in the circuit is

$$p = ei = E_0 I_0 \sin^2 \theta = \frac{E_0 I_0}{2}(1 - \cos 2\theta). \qquad (169)$$

The values of e, i and p for a complete cycle are plotted in Fig. 81.

The power varies with twice the frequency of the current from O to E_0I_0, but never becomes negative. The area beneath the power curve represents the energy consumed in the circuit during one cycle.

The average power is

$$P = \frac{1}{\pi}\int_0^\pi p\,d\theta$$

$$= \frac{1}{\pi}\int_0^\pi \frac{E_0I_0}{2}(1 - \cos 2\,\theta)\,d\theta$$

$$= \frac{E_0I_0}{2\,\pi}\left[\theta - \frac{\sin 2\,\theta}{2}\right]$$

$$= \frac{E_0I_0}{2\,\pi}[\pi]$$

$$= \frac{E_0I_0}{2} = \frac{E_0}{\sqrt{2}}\cdot\frac{I_0}{\sqrt{2}} = EI; \quad \ldots \quad (170)$$

therefore, the average power in an alternating-current circuit is equal to the product of the effective values of the e.m.f. and the current, if they are in phase.

Since

$$P = EI$$

and

$$E = IR,$$

therefore,

$$P = I^2R, \quad \ldots \quad (171)$$

and the power is equal to the square of the effective value of the current multiplied by the resistance.

Case II. If an e.m.f. $e = E_0 \sin \theta$ is impressed on a circuit containing a resistance R, and an inductive reactance X, the current will lag behind the e.m.f. by an angle $\phi = \tan^{-1}\frac{X}{R}$ and its instantaneous value will be

$$i = I_0 \sin (\theta - \phi).$$

The instantaneous power in the circuit is

$$p = ei = E_0I_0 \sin \theta \sin (\theta - \phi)$$

$$= \frac{E_0I_0}{2}\{\cos \phi - \cos (2\,\theta - \phi)\}. \quad . \quad (172)$$

The values of e, i and p for one cycle are plotted in Fig. 82.

The maximum value of power is $\frac{E_0 I_0}{2}(1 + \cos\phi)$, which is less than in Case I, and the power curve falls below the base twice in

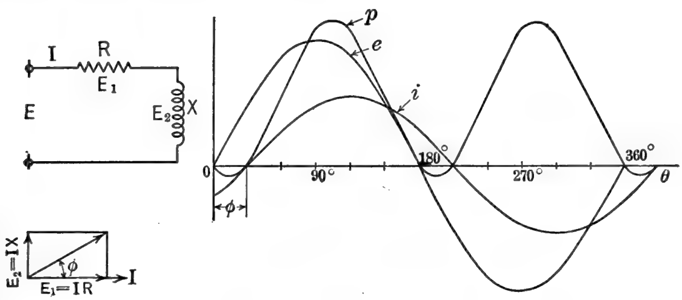


FIG. 82. Power in an inductive circuit.

each cycle. The energy consumed per cycle is the difference between the positive and negative areas intercepted by the power curve.

The average power is

$$P = \frac{1}{\pi}\int_0^\pi p\,d\theta$$

$$= \frac{E_0 I_0}{2\pi}\int_0^\pi \{\cos\phi - \cos(2\theta - \phi)\}\,d\theta$$

$$= \frac{E_0 I_0}{2\pi}\left[\theta \cdot \cos\phi - \frac{\sin(2\theta - \phi)}{2}\right]_0^\pi$$

$$= \frac{E_0 I_0}{2\pi}\left[\pi \cdot \cos\phi - \frac{\sin(-\phi)}{2} + \frac{\sin(-\phi)}{2}\right]$$

$$= \frac{E_0 I_0}{2}\cos\phi$$

$$= EI\cos\phi; \qquad (173)$$

therefore, the average power is the product of the effective values of the e.m.f. and current multiplied by the cosine of the angle of phase difference between them.

From the vector diagram, in Fig. 82, it is seen that

$$E\cos\phi = E_1 = IR,$$

and therefore the power is

$$P = E_1 I = I^2 R,$$

and is equal to the in-phase component of the e.m.f. multiplied by the current or is equal to the square of the current multiplied by the resistance of the circuit as found in equation 171; thus, all the power consumed in the circuit is consumed by the resistance. Reactance or self-inductance does not consume power since the energy stored while the current is increasing is given back while

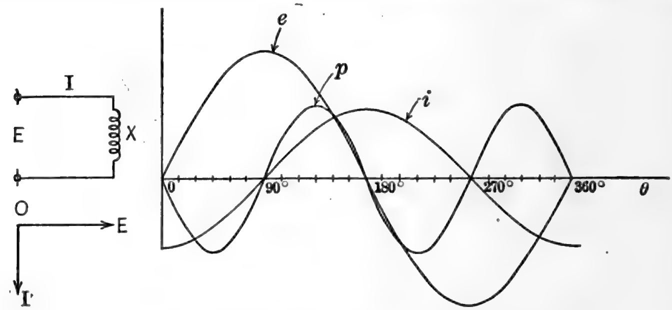


Fig. 83. Power in an inductive reactance.

it is decreasing and the e.m.f. consumed by self-inductance is a wattless e.m.f. Similarly condensive reactance does not consume power since the energy stored while the e.m.f. is increasing is given back to the circuit while the e.m.f. is decreasing and the e.m.f. consumed by condensive reactance is wattless.

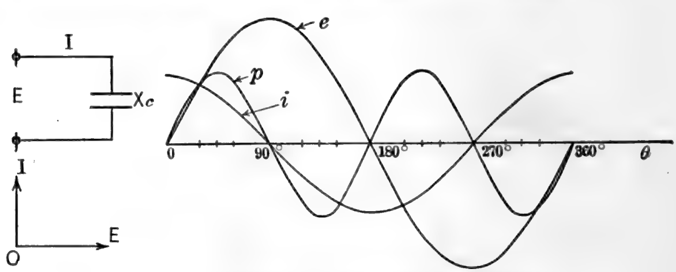


Fig. 84. Power in a condensive reactance.

These results are illustrated in Fig. 83 and Fig. 84.

In Fig. 83 are plotted the values of e, i and p for an inductive circuit without resistance in which the current lags 90 degrees behind the impressed e.m.f. The power curve cuts off equal areas above and below the base line and therefore the average power is zero. The area below the line represents the energy

given back by the magnetic field while the current is decreasing from its maximum value I_0 to zero and the area above the line represents the energy stored in the magnetic field while the current is increasing again to its maximum value. The amount of energy in each case is $L\frac{I_0^2}{2}$ watt-seconds.

In Fig. 84 are plotted the values of e, i and p for a circuit containing a condensive reactance but without resistance. The current leads the impressed e.m.f. by 90 degrees and the average power is again zero, so that no energy is consumed in the circuit. The positive area cut off by the power curve represents the energy stored in the electrostatic field of the condenser while the e.m.f. is increasing and the negative area represents the energy returned to the circuit while the e.m.f. is decreasing. The maximum amount of energy in each case is $C\frac{E_0^2}{2}$ watt-seconds where E_0 is the maximum e.m.f.

Fig. 85 illustrates various methods of representing the power in a circuit; in (a) it is the product of the impressed e.m.f., the

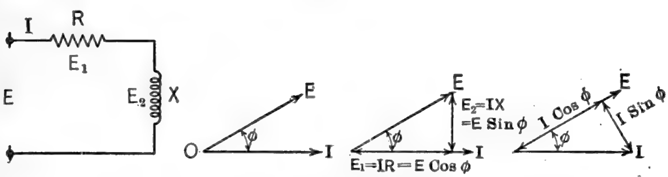


FIG. 85. Power in alternating-current circuits.

current and the cosine of the angle of phase difference between them,

$$P = E \times I \times \cos\phi; \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

in (b) it is the product of the current and the in-phase component of the e.m.f.,

$$P = I \times E\cos\phi = I \times E_1 = I^2R; \quad . \quad . \quad . \quad (175)$$

in (c) it is the product of the e.m.f. and the in-phase component of the current,

$$P = E \times I\cos\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

The apparent power in a circuit is the product of the impressed e.m.f. and the current and is expressed in volt amperes or kilovolt amperes.

The power factor of a circuit is the ratio of the true power to the apparent power and is

$$\frac{P}{EI} = \frac{EI \cos \phi}{EI} = \cos \phi; \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

therefore, the power factor is the cosine of the angle of phase difference between the current and the impressed e.m.f.; it is usually expressed in per cent and may be either leading or lagging.

When the current is in phase with the e.m.f., the power factor is unity or 100 per cent.

When the current leads the e.m.f. by 60 degrees the power factor is 50 per cent leading, since $\cos 60° = 0.5$; when the current lags behind the e.m.f. by 60 degrees the power factor is 50 per cent lagging.

The sine of the angle of phase difference between the current and the impressed e.m.f. is called the inductance factor of the circuit.

81. Examples. (1) If an alternating e.m.f. of effective value E is impressed on a non-inductive circuit of resistance R, a current I will flow in phase with the e.m.f., where

$$I = \frac{E}{R}.$$

The vector diagram is shown in Fig. 86.

(2) If an alternating e.m.f. E is impressed on a circuit of reactance X and negligible resistance, a current $I = \frac{E}{X}$ will flow lagging 90 degrees behind the e.m.f. (See Fig. 87.)

(3) If an alternating e.m.f. E is impressed on a circuit of resistance R and reactance X, a current I will flow lagging behind the e.m.f. by angle ϕ, where

$$\tan \phi = \frac{X}{R}.$$

The vector diagram for the circuit is shown in Fig. 88. The e.m.f. consumed in the resistance is $E_1 = IR$ volts in phase with the current and is represented by the vector OE_1.

The e.m.f. consumed by the reactance is $E_2 = IX$ volts leading the current by 90 degrees represented by OE_2. The impressed e.m.f. is $OE = E$ and is the vector sum of E_1 and E_2; therefore

$$E = \sqrt{E_1^2 + E_2^2} = \sqrt{\overline{IR}^2 + \overline{IX}^2} = I\sqrt{R^2 + X^2} = IZ,$$

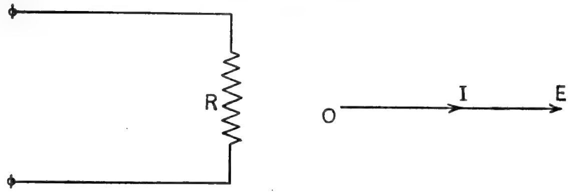


FIG. 86.

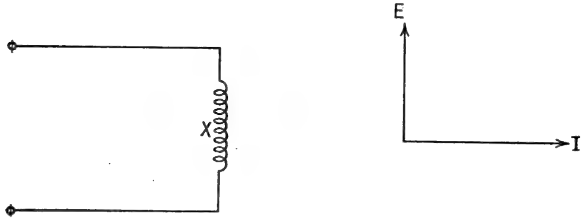


FIG. 87.

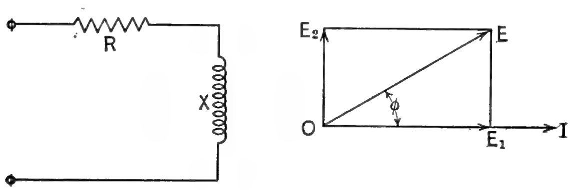


FIG. 88.

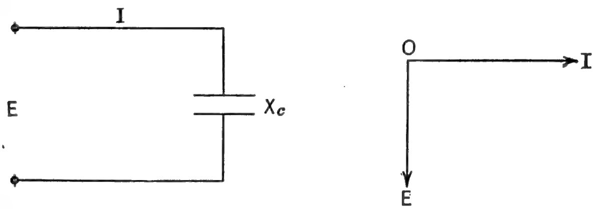


FIG. 89.

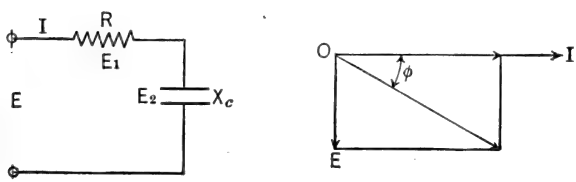


FIG. 90.

where

$Z = \sqrt{R^2 + X^2}$ is the impedance of the circuit.

(4) Fig. 89 shows the vector diagram of a circuit containing a condenser of reactance X_c, when an alternating e.m.f. E is impressed on its terminals and a current $I = \frac{E}{X_c}$ flows through it leading the e.m.f. by 90 degrees.

(5) Fig. 90 shows the diagram for the same circuit with a resistance R added in series.

The e.m.f. consumed by the resistance is $OE_1 = E_1 = IR$, in phase with $OI = I$.

The e.m.f. consumed by the reactance is $OE_2 = E_2 = IX_c$, lagging 90 degrees behind OI.

The impressed e.m.f. is

$$OE = E = \sqrt{E_1^2 + E_2^2} = I\sqrt{R^2 + X_c^2} = IZ,$$

lagging behind OI by angle ϕ, where

$$\tan \phi = \frac{E_2}{E_1} = \frac{IX_c}{IR} = \frac{X_c}{R}.$$

(6) If an alternating e.m.f. E is impressed on a circuit containing a resistance R, an inductive reactance $X = 2\pi fL$ and a condensive reactance $X_c = \frac{1}{2\pi fC}$ connected in series, determine the magnitude and phase relation of the current and draw the vector diagram for the circuit. (Fig. 91.)

$OI = I$ is the current taken as horizontal.

$OE_1 = E_1 = IR$ is the e.m.f. consumed by the resistance and is in phase with I.

$OE_2 = E_2 = IX$ is the e.m.f. consumed by the inductive reactance and leads OI by 90 degrees.

$OE_3 = E_3 = IX_c$ is the e.m.f. consumed by the condensive reactance and lags behind OI by 90 degrees.

$OE = E = IZ$ is the e.m.f. impressed on the circuit, or the e.m.f. consumed by the impedance Z. It is the vector sum of the three components E_1, E_2 and E_3 and leads the current by angle ϕ; therefore,

$$\begin{aligned} E &= \sqrt{E_1^2 + (E_2 - E_3)^2} \\ &= I\sqrt{R^2 + (X - X_c)^2} \\ &= IZ, \end{aligned}$$

and

$$Z = \sqrt{R^2 + (X - X_c)^2}$$

$$= \sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2},$$

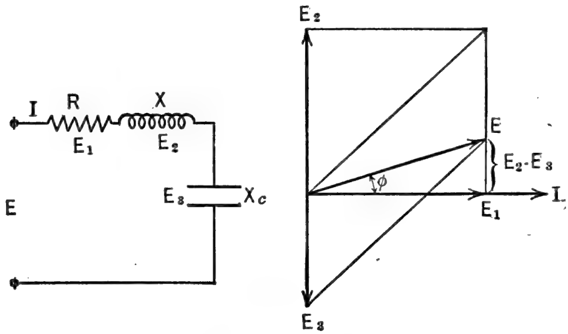


FIG. 91.

where L is the inductance of the circuit in henrys, C is the capacity in farads and f is the frequency of the impressed e.m.f.

The angle of phase difference between the e.m.f. and current is ϕ, where

$$\tan\phi = \frac{E_2 - E_3}{E_1} = \frac{I(X - X_c)}{IR} = \frac{X - X_c}{R}.$$

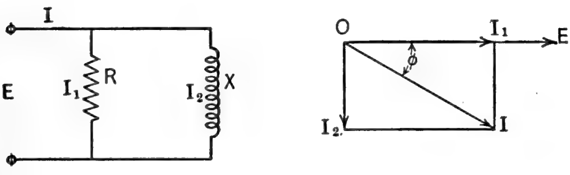


FIG. 92.

If $X > X_c$ the current lags behind the e.m.f.;
if $X = X_c$ the current is in phase with the e.m.f.;
and if $X < X_c$ the current leads the e.m.f.

(7) An alternating e.m.f. E is impressed on the terminals of the circuit in Fig. 92 consisting of a resistance R and an inductive reactance X in parallel.

The main current I has two components,

$$I_1 = \frac{E}{R}, \text{ in phase with } E,$$

and

$$I_2 = \frac{E}{X}, \text{ 90 degrees behind } E.$$

From the vector diagram

$$I = \sqrt{I_1^2 + I_2^2} = E\sqrt{\frac{1}{R^2} + \frac{1}{X^2}} = E\frac{\sqrt{R^2 + X^2}}{RX}$$

and lags behind E by an angle ϕ, where

$$\tan\phi = \frac{I_2}{I_1} = \frac{E/X}{E/R} = \frac{R}{X}.$$

The impedance of the circuit is

$$Z = \frac{E}{I} = \frac{RX}{\sqrt{R^2 + X^2}}.$$

(8) In Fig. 93 a third branch is connected in parallel with the

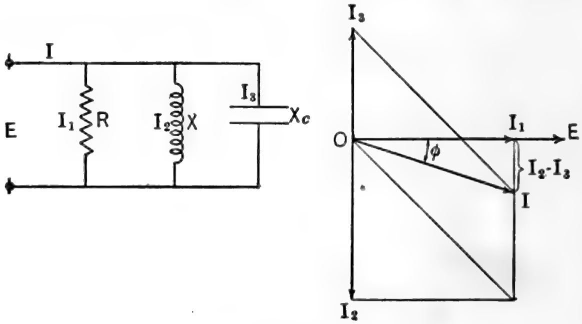


Fig. 93.

two in example (7) containing a condensive reactance $X_c < X$.

The main current I has three components,

$$I_1 = \frac{E}{R}, \text{ in phase with } E,$$

$$I_2 = \frac{E}{X}, \text{ 90 degrees behind } E,$$

and

$$I_3 = \frac{E}{X_c}, \text{ 90 degrees ahead of } E.$$

From the vector diagram

$$I = \sqrt{I_1^2 + (I_2 - I_3)^2} = E\sqrt{\frac{1}{R^2} + \left(\frac{1}{X} - \frac{1}{X_c}\right)^2}$$

and leads E by an angle ϕ, where

$$\tan\phi = \frac{I_3 - I_2}{I_1} = \frac{1/X_c - 1/X}{1/R}.$$

The impedance of the circuit is

$$Z = \frac{E}{I} = \frac{1}{\sqrt{\frac{1}{R^2} + \left(\frac{1}{X} - \frac{1}{X_c}\right)^2}}.$$

(9) Find the magnitudes of the currents in the various parts of the circuit in Fig. 94 and their phase relations with the impressed e.m.f.

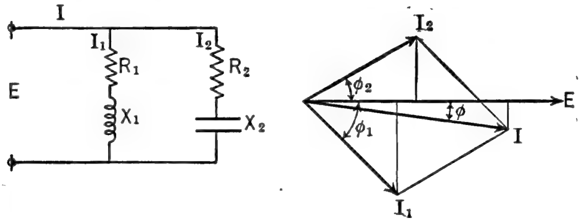


FIG. 94.

$I_1 = \dfrac{E}{\sqrt{R_1^2 + X_1^2}}$ and lags behind the impressed e.m.f. by an angle ϕ_1, where $\tan\phi_1 = \dfrac{X_1}{R_1}$;

$I_2 = \dfrac{E}{\sqrt{R_2^2 + X_2^2}}$ and leads the e.m.f. by an angle ϕ_2, where $\tan\phi_2 = \dfrac{X_c}{R_2}$;

$I = \sqrt{I_1^2 + I_2^2 + 2\,I_1 I_2 \cos(\phi_1 + \phi_2)}$ (see vector diagram) and lags behind the e.m.f. by an angle ϕ, where

$$\tan\phi = \frac{I_1\sin\phi_1 - I_2\sin\phi_2}{I_1\cos\phi_1 + I_2\cos\phi_2}.$$

The power consumed in the circuit is

$$P = EI\cos\phi$$
$$= I_1^2R_1 + I_2^2R_2 \text{ watts.}$$

82. Numerical Examples. (1) If an alternating e.m.f. of 200 volts at a frequency of 60 cycles per second is impressed on a circuit consisting of a resistance of 10 ohms in series with an inductance of 0.1 henry and a capacity of 100 microfarads, (a) determine the current in the circuit and its phase relation with the impressed e.m.f., (b) the e.m.f. consumed in each part of the circuit. (c) If the impressed e.m.f is maintained constant and the frequency is varied determine the maximum value of the current.

(a) The inductive reactance of the circuit is

$$X = 2\pi f L = 2 \times 3.14 \times 60 \times 0.1 = 37.6 \text{ ohms;}$$

the condensive reactance is

$$X_c = \frac{1}{2\pi f C} = \frac{10^6}{2 \times 3.14 \times 60 \times 100} = 26.4 \text{ ohms;}$$

the impedance of the circuit is

$$Z = \sqrt{R^2 + (X - X_c)^2} = \sqrt{10^2 + (37.6 - 26.4)^2} = 15 \text{ ohms;}$$

and therefore the current is

$$I = \frac{E}{Z} = \frac{200}{15} = 13.3 \text{ amperes.}$$

The current lags behind the e.m.f. by an angle ϕ, where

$$\tan \phi = \frac{X - X_c}{R} = \frac{37.6 - 26.4}{10} = 1.12$$

and

$$\phi = 48°18'.$$

(b) The e.m.f. consumed in the resistance is

$$E_1 = IR = 13.3 \times 10 = 133 \text{ volts;}$$

the e.m.f. consumed in the inductive reactance is

$$E_2 = IX = 13.3 \times 37.6 = 500 \text{ volts;}$$

and the e.m.f. consumed by the condensive reactance is

$$E_3 = IX_c = 20 \times 26.4 = 350 \text{ volts.}$$

The impressed e.m.f. is

$$E = \sqrt{E_1^2 + (E_2 - E_3)^2} = \sqrt{133^2 + (500 - 350)^2} = 200 \text{ volts.}$$

The vector diagram is shown in Fig. 95.

(c) The current in the circuit at any frequency is

$$I = \frac{E}{Z} = \frac{E}{\sqrt{R^2 + \left((2\pi f L - \frac{1}{2\pi f C}\right)^2}}.$$

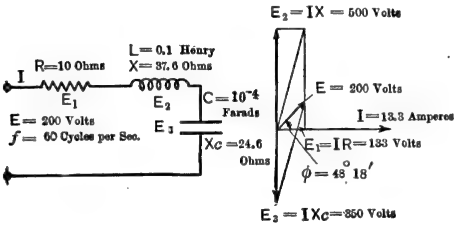


FIG. 95.

If E is maintained constant at 200 volts and f is varied, I varies. When $f = 0$, $\frac{1}{2\pi fC} = \propto$ and $I = 0$;

when $f = \propto$, $2\pi fL = \propto$ and $I = 0$;

when $2\pi fL = \frac{1}{2\pi fC}$ or $f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2 \times 3.14\sqrt{.1 \times \frac{100}{10^6}}}$

$$= 50 \text{ cycles}$$

the current has its maximum value

$$I_{\text{max.}} = \frac{E}{R} = \frac{200}{10} = 20 \text{ amperes},$$

and is in phase with the impressed e.m.f.

The e.m.f. consumed in the resistance is

$$E_1 = IR = 20 \times 10 = 200 \text{ volts};$$

the e.m.f. consumed in the inductive reactance is

$$E_2 = IX = 20 \times 2 \times 3.14 \times 50 \times 0.1 = 628 \text{ volts};$$

and the e.m.f. consumed in the condensive reactance is

$$E_3 = IX_c = 20 \times \frac{1}{2 \times 3.14 \times 50 \times \frac{100}{10^6}} = 628 \text{ volts}.$$

The vector diagram for the circuit is shown in Fig. 96, and the current and e.m.f. waves are shown in Fig. 97. A series circuit in which the inductive reactance and the condensive reactance are equal at a certain frequency is said to be in a state of resonance for that frequency. In commercial circuits the capacity is usually so small that resonance cannot occur at ordinary frequencies, but when any high frequency e.m.f.'s are produced in the circuit resonance may occur and very large e.m.f.'s may appear and break down the insulation.

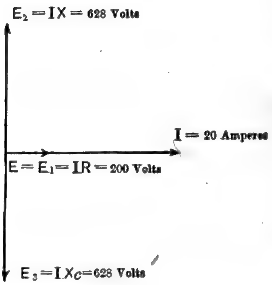


FIG. 96. Resonant circuit.

(2) If an alternating e.m.f. $E = 200$ volts at a frequency $f = 60$ cycles per second is impressed on the circuit in Fig. 98, determine the value and phase relation of the main current and the currents in the three branches.

The first branch is a resistance $R = 40$ ohms; the second branch is an inductance $L = 0.1$ henry and has a reactance $X = 2\pi fL = 37.6$ ohms at 60 cycles; the third branch is a capacity $C = 100$ microfarads or 10^{-4} farads and has a reactance $X_c = \dfrac{1}{2\pi fC} = 26.4$ ohms.

The current in the resistance is

$$I_1 = \frac{E}{R} = \frac{200}{40} = 5 \text{ amperes},$$

in phase with the impressed e.m.f.; the current in the inductive reactance is

$$I_2 = \frac{E}{X} = \frac{200}{37.6} = 5.3 \text{ amperes},$$

90 degrees behind the impressed e.m.f.; the current in the condensive reactance is

$$I_3 = \frac{E}{X_c} = \frac{200}{26.4} = 7.6 \text{ amperes},$$

90 degrees ahead of the impressed e.m.f.

The main current is

$$I = \sqrt{I_1^2 + (I_2 - I_3)^2} = \sqrt{5^2 + (5.3 - 7.6)^2} = 5.5 \text{ amperes},$$

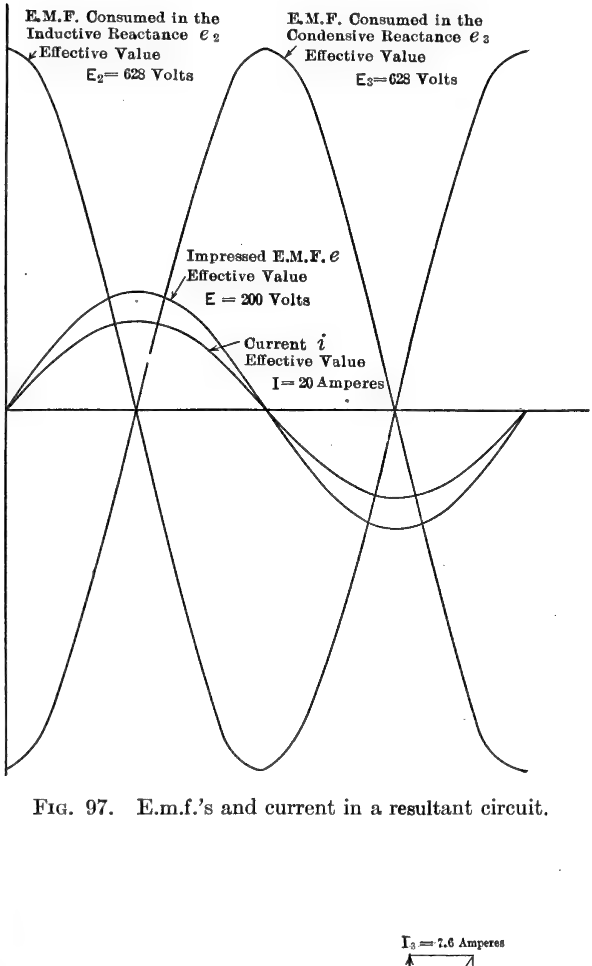


FIG. 97. E.m.f.'s and current in a resultant circuit.

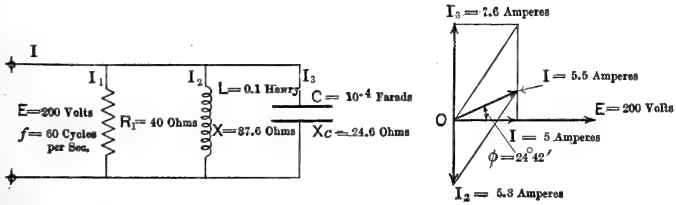


FIG. 98.

leading the impressed e.m.f. by an angle ϕ, where

$$\tan \phi = \frac{I_3 - I_2}{I_1} = \frac{2.3}{5} = 0.46,$$

and therefore

$$\phi = 24° 42'.$$

If the e.m.f. impressed on the circuit is maintained constant and the frequency is varied find the magnitude of the main current when it is in phase with the e.m.f.

The current at any frequency is

$$I = \sqrt{I_1^2 + (I_2 - I_3)^2}$$

$$= E\sqrt{\frac{1}{R^2} + \left(\frac{1}{X} - \frac{1}{X_c}\right)^2}$$

$$= E\sqrt{\frac{1}{R^2} + \left(\frac{1}{2\pi fL} - 2\pi fC\right)^2};$$

the angle of lag of the current behind the e.m.f. is

$$\phi = \tan^{-1}\frac{I_2 - I_3}{I_1} = \tan^{-1}\frac{1/X - 1/X_2}{1/R}.$$

When the current is in phase with the e.m.f.

$$\phi = 0 \text{ and } X = X_c \text{ or } 2\pi fL = \frac{1}{2\pi fC};$$

the frequency is therefore

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2 \times 3.14\sqrt{.1 \times 10^{-4}}} = 50 \text{ cycles.}$$

The main current at a frequency of 50 cycles is

$$I = I_1 = \frac{E}{R} = \frac{200}{40} = 5 \text{ amperes};$$

the current in the inductive reactance is

$$I_2 = \frac{E}{X} = \frac{200}{2 \times 3.14 \times 50 \times 0.1} = 6.36 \text{ amperes};$$

the current in the condensive reactance is

$$I_3 = \frac{E}{X} = \frac{200}{\dfrac{1}{2 \times 3.14 \times 50 \times 10^{-4}}} = 6.36 \text{ amperes};$$

and the current in the lead between the first and second branches is zero.

(3) If an alternating e.m.f. E of the wave shape shown in Fig. 99 (a) is impressed on the terminals of the circuit AB find the currents in the various branches.

The e.m.f. wave consists of a fundamental sine wave of effective value 100 volts at 50 cycles and a fifth harmonic of effective value 10 volts.

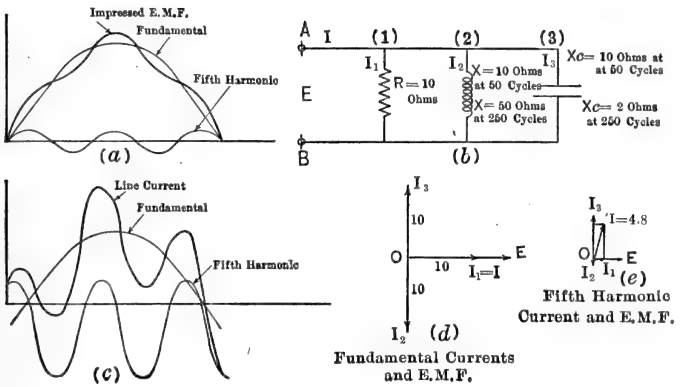


FIG. 99.

The resistance of branch (1) is 10 ohms; the reactance of branch (2) is 10 ohms at 50 cycles and 50 ohms at 250 cycles; the condensive reactance of branch (3) is 10 ohms at 50 cycles and 2 ohms at 250 cycles.

The current I_1 in (1) consists of a fundamental of $\frac{100}{10} = 10$ amperes and a fifth harmonic of $\frac{10}{10} = 1$ ampere in phase with their respective e.m.f.'s.

The wave shape of the current in the resistance is the same as that of the impressed e.m.f.

The current I_2 in (2) consists of a fundamental of $\frac{100}{10} = 10$ amperes and a fifth harmonic of $\frac{10}{50} = 0.2$ ampere in quadrature behind the e.m.f.'s producing them. The fifth harmonic is not nearly so prominent as in the resistance circuit and thus reactance tends to smooth out irregular waves and make them more nearly approximate to sine waves.

The current I_3 in (3) consists of a fundamental of $\frac{100}{10} = 10$

amperes and a fifth harmonic of $\frac{10}{2} = 5$ amperes in quadrature ahead of the e.m.f.'s producing them. The fifth harmonic is much more prominent in the capacity circuit than in either of the others and thus the capacity tends to exaggerate the harmonics in a peaked wave.

The main current I consists of a fundamental and a fifth harmonic. The fundamental is the resultant of the fundamental currents in the three branches and from Fig. 99 (d) is found to be 10 amperes in phase with fundamental e.m.f.

The fifth harmonic is the resultant of the fifth harmonic currents in the three branches and from Fig. 99 (e) it is found to be 4.8 amperes, leading the fifth harmonic e.m.f. by nearly a quarter of a cycle. The main current is shown in Fig. 99 (c).

(4) If the same e.m.f. is impressed on the terminals of the circuit CD in Fig. 100 (a) determine the current flowing. The re-

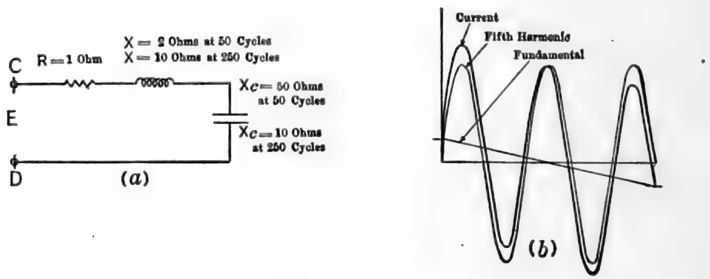


Fig. 100.

sistance of the circuit is 1 ohm, the reactance is 2 ohms at 50 cycles and 10 ohms at 250 cycles and the condensive reactance is 50 ohms at 50 cycles and 10 ohms at 250 cycles.

The main current consists of a fundamental of $\frac{100}{\sqrt{1 + (50 - 2)^2}}$ $= \frac{100}{48} = 2.05$ amperes leading the fundamental e.m.f. by nearly 90 degrees and a fifth harmonic of $\frac{10}{\sqrt{1^2 + (10 - 10)^2}} = 10$ amperes.

The circuit is resonant for the fifth harmonic e.m.f. at 250 cycles and so the fifth harmonic current is very much exaggerated. The current is plotted in Fig. 100 (b).

83. Circuit Constants. A continuous-current circuit has two constants,

$$\text{resistance } R, r = \frac{\text{impressed e.m.f.}}{\text{current}},$$

$$\text{conductance } G, g = \frac{\text{current}}{\text{impressed e.m.f.}},$$

and the conductance is the reciprocal of the resistance or

$$G = \frac{1}{R}.$$

Continuous-current circuits also have inductance and electrostatic capacity, but these do not affect the flow of current except at the instant of opening or closing the circuit.

An alternating-current circuit has six so-called constants,

(1) resistance R, r,
(2) reactance X, x,
(3) impedance Z, z,
(4) admittance Y, y,
(5) conductance G, g,
(6) susceptance B, b.

(1) The resistance of a circuit consumes a component of e.m.f. in phase with the current and so consumes power. In circuits which are partially inclosed in iron an alternating magnetic flux is produced in the iron and a loss of power occurs due to hysteresis and eddy currents. These iron losses are sometimes included with the copper loss and charged against the resistance. This gives a value of resistance greater than the true ohmic resistance and is called the effective resistance of the circuit. Since the hysteresis and eddy current losses vary both with the frequency and the induction density in the iron, the effective resistance is not a constant quantity.

The power component of the impressed e.m.f. or the component in phase with the current is

$$E_1 = IR,$$

and the resistance is

$$R = \frac{E_1}{I} = \frac{\text{power component of impressed e.m.f.}}{\text{current}}$$

$$= \frac{\text{in-phase component of impressed e.m.f.}}{\text{current}} \quad . \quad . \quad (178)$$

(2) The reactance of a circuit consumes a component of e.m.f. in quadrature with the current, leading in the case of circuits of large inductance and lagging in circuits of large electrostatic capacity, but it does not consume any power.

The inductive reactance of a circuit is

$$X = 2\pi f L \text{ ohms,}$$

where f is the frequency of the impressed e.m.f. and L is the inductance in henrys. Commercial circuits are operated at a fixed frequency and so f is constant.

The inductance of a circuit in air or any non-magnetic material is constant but in an iron-clad circuit it varies with the current, decreasing as the current increases since the permeability of the iron decreases as the flux density in it increases.

Since inductive reactance consumes a component of e.m.f. in quadrature ahead of the current it is taken as a positive reactance.

The condensive reactance of a circuit is

$$X_C = \frac{1}{2\pi f C} \text{ ohms,}$$

where C is the capacity of the circuit in farads. The capacity of a circuit does not vary with the current or e.m.f. and thus the condensive reactance is constant so long as the frequency is constant.

Condensive reactance is taken as a negative reactance since it consumes a component of e.m.f. in quadrature behind the current. Thus when inductive reactance and condensive reactance are connected in series they oppose and the reactance of the circuit is

$$X = X_L - X_C = 2\pi f L - \frac{1}{2\pi f C} \text{ ohms.}$$

In series-parallel circuits the reactance is a complex function of the resistances and reactances of the various branches.

The reactance of any circuit is

$$X = \frac{\text{wattless component of impressed e.m.f.}}{\text{current}}$$

$$= \frac{\text{quadrature component of impressed e.m.f.}}{\text{current.}} \quad . \; . \quad (179$$

(3) The impedance of a circuit includes both the resistance and the reactance; it is

$$Z = \sqrt{R^2 + X^2} = \frac{\text{impressed e.m.f.}}{\text{current}}. \quad . \quad . \quad (180)$$

(4) The admittance of a circuit is

$$Y = \frac{\text{current}}{\text{impressed e.m.f.}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

it is the reciprocal of the impedance and thus

$$Y = \frac{1}{Z} = \frac{1}{\sqrt{R^2 + X^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

The admittance has two components, conductance and susceptance.

Fig. 101 shows a circuit of impedance $Z = \sqrt{R^2 + X^2}$ in which the current lags behind the impressed e.m.f. by an angle

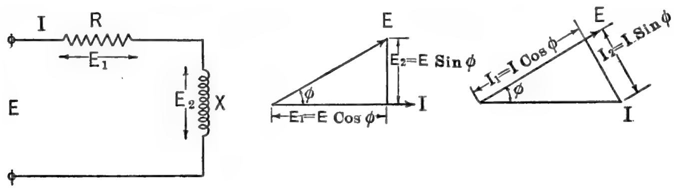


FIG. 101.

ϕ. The current I can be resolved into two components in phase and in quadrature with the impressed e.m.f. E. The in-phase or power component of current is

$$I_1 = I \cos \phi = \frac{E}{Z} \cos \phi = E \frac{R}{Z^2} = EG, \quad . \quad . \quad . \quad (183)$$

where

$$G = \frac{R}{Z^2} = \frac{R}{R^2 + X^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

is the conductance of the circuit.

The quadrature or wattless component of current is

$$I_2 = I \sin \phi = \frac{E}{Z} \sin \phi = E \frac{X}{Z^2} = EB, \quad . \quad . \quad (185)$$

where $$B = \frac{X}{Z^2} = \frac{X}{R^2 + X^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

is the susceptance of the circuit.

The total current

$$I = \sqrt{I_1^2 + I_2^2} = \sqrt{EG^2 + EB^2} = E\sqrt{G^2 + B^2},$$

but, by equation 181, $I = EY$, and therefore

$$Y = \sqrt{G^2 + B^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

(5) The conductance of a circuit is, from equation 184,

$$G = \frac{\text{power component of current}}{\text{impressed e.m.f}}. \quad . \quad . \quad . \quad (188)$$

(6) The susceptance of a circuit is, from equation 185,

$$B = \frac{\text{wattless component of current}}{\text{impressed e.m.f.}}. \quad . \quad . \quad (189)$$

In the solution of series circuits it is not necessary to employ the terms admittance, conductance and susceptance but the solution of series-parallel circuits is very much simplified by their use.

84. Example. In the circuit in Fig. 102 determine the main current and the currents in the two branches in magnitude and phase relation with the impressed e.m.f.

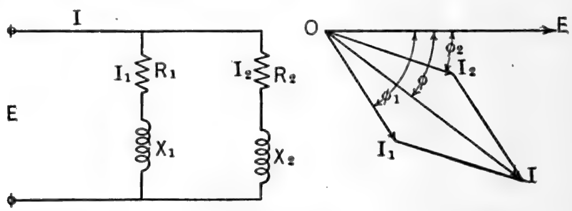


FIG. 102.

(I) Using the constants R and X:

(1) $$I_1 = \frac{E}{\sqrt{R_1^2 + X_1^2}}, \qquad \tan \phi_1 = \frac{X_1}{R_1};$$

(2) $$I_2 = \frac{E}{\sqrt{R_2^2 + X_2^2}}, \qquad \tan \phi_2 = \frac{X_2}{R_2};$$

from the vector diagram

(3) $$I = \sqrt{I_1^2 + I_2^2 + 2\,I_1 I_2 \cos \phi}$$

and

(4) $$\tan \phi = \frac{I_1 \sin \phi_1 + I_2 \sin \phi_2}{I_1 \cos \phi_1 + I_2 \sin \phi_2}.$$

Substituting values obtained from (1) and (2) in the equations (3) and (4) the magnitude and phase relation of I can be obtained.

(II) Using the constants G and B:

$$(1) \qquad I_1 = E\sqrt{G_1^2 + B_1^2}, \qquad \tan\phi_1 = \frac{B_1}{G_1};$$

$$(2) \qquad I_2 = E\sqrt{G_2^2 + B_2^2}, \qquad \tan\phi_2 = \frac{B_2}{G_2}.$$

From the vector diagram

$$I = E\sqrt{(G_1 + G_2)^2 + (B_1 + B_2)^2}$$

and $$\tan\phi = \frac{B_1 + B_2}{G_1 + G_2}.$$

85. Rectangular Coördinates. The simplest method of dealing with alternating-current phenomena is to express the e.m.f.'s, currents, etc., as the sum of two components, one along a chosen axis and the other perpendicular to it.

In Fig. 103 which represents the e.m.f. and current in a circuit of impedance $z = \sqrt{r^2 + x^2}$ the axis is chosen in the direction of

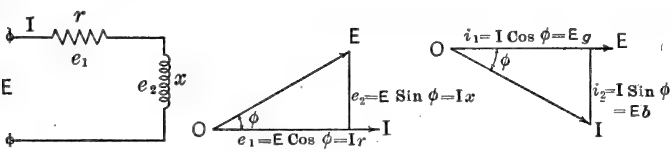


FIG. 103.

the current and the e.m.f. is resolved into two components, e_1 in phase with the current and e_2 in quadrature ahead of the current.

The absolute value of the e.m.f. is

$$E = \sqrt{e_1^2 + e_2^2},$$

and it leads the current which was chosen as axis by an angle ϕ,

where $$\tan\phi = \frac{e_2}{e_1}.$$

Thus when the e.m.f. is expressed as the sum of two components at right angles both its magnitude and its phase are known.

To distinguish between horizontal and vertical components the prefix $j = \sqrt{-1}$ is added to all vertical components and the expression for the e.m.f. above is

$$\underset{\cdot}{E} = e_1 + je_2. \qquad (190)$$

The dot is placed under the E to show that it is expressed in rectangular coördinates and serves to distinguish it from its absolute value.

Since $e_1 = E \cos \phi = Ir$ and $e_2 = E \sin \phi = Ix$,

$$\begin{aligned} \underset{\cdot}{E} &= E \cos \phi + jE \sin \phi \\ &= Ir + jIx \\ &= I\,(r + jx), \qquad (191) \end{aligned}$$

and therefore the impedance in rectangular coördinates is

$$\underset{\cdot}{z} = r + jx. \qquad (192)$$

In Fig. 103 the e.m.f. is chosen as axis and the current is behind it in phase by an angle ϕ and has two components i_1 in phase with the e.m.f. and i_2 in quadrature behind it.

The current may be written

$$\underset{\cdot}{I} = i_1 - ji_2, \qquad (193)$$

and this equation indicates that the current has a value

$$I = \sqrt{i_1^2 + i_2^2},$$

and that it is behind the chosen axis (in this case the e.m.f.) in phase by an angle ϕ, where

$$\tan \phi = \frac{i_2}{i_1}.$$

Since $i_1 = I \cos \phi = Eg$ and $i_2 = I \sin \phi = Eb$, where $\underset{\cdot}{y} = \sqrt{g^2 + b^2}$ is the admittance of the circuit, equation 193 may be written

$$\begin{aligned} \underset{\cdot}{I} &= I \cos \phi - jI \sin \phi \\ &= Eg - jEb \\ &= E\,(g - jb), \qquad (194) \end{aligned}$$

and therefore the admittance in rectangular coördinates is

$$\underset{\cdot}{y} = g - jb.$$

Admittance and impedance are not alternating quantities and their components are independent of the axis of reference but they can be represented in rectangular coördinates as shown.

A current multiplied by an impedance gives an e.m.f. displaced from it in phase by an angle whose tangent is the ratio of the reactance to the resistance.

A current divided by an admittance gives an e.m.f. displaced in phase by an angle whose tangent is the ratio of the susceptance to the conductance.

Similarly an e.m.f. divided by an impedance or multiplied by an admittance gives a current.

By definition a vector multiplied by j is turned through 90 degrees in the counter-clockwise direction; when multiplied by j^2 it is turned through 180 degrees and its sign is reversed.

Therefore $$j^2 = -1$$

and $$j = \sqrt{-1}. \qquad (195)$$

Taking this value for j alternating quantities expressed in rectangular coördinates referred to a given axis can be added, subtracted, multiplied and divided and the results obtained are expressed in rectangular coördinates referred to the same axis.

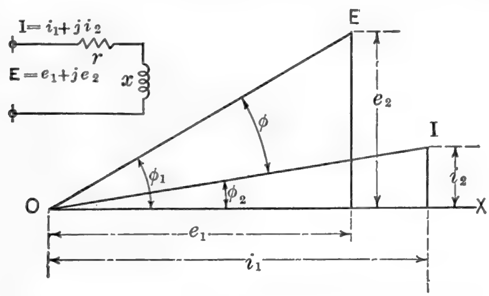


FIG. 104. Power in rectangular coördinates.

It is not necessary to choose either the current in a circuit or the e.m.f. as axis and any other line may be taken as shown in Fig. 104, but the e.m.f must now be expressed as the sum of two components along and perpendicular to the new axis,

$$\underset{\cdot}{E} = e_1 + je_2,$$

and similarly the current is

$$\underset{\cdot}{I} = i_1 + ji_2.$$

The e.m.f. has an absolute value

$$E = \sqrt{e_1^2 + e_2^2}$$

and is ahead of the axis by an angle

$$\phi_1 = \tan^{-1}\frac{e_2}{e_1} = \cos^{-1}\frac{e_1}{E}.$$

The current has an absolute value

$$I = \sqrt{I_1^2 + i_2^2}$$

and is ahead of the axis by an angle

$$\phi_2 = \tan^{-1}\frac{i_2}{i_1} = \cos^{-1}\frac{i_1}{I}.$$

The e.m.f. leads the current by an angle

$$\phi = \phi_1 - \phi_2.$$

The impedance of the circuit in rectangular coördinates is

$$\begin{aligned} z = \frac{E}{I} &= \frac{e_1 + je_2}{i_1 + ji_2} = \frac{e_1 + je_2}{i_1 + ji_2} \times \frac{i_1 - ji_2}{i_1 - ji_2} \\ &= \frac{e_1 i_1 + e_2 i_2}{i_1^2 + i_2^2} + j\frac{e_2 i_1 - e_1 i_2}{i_1^2 + i_2^2} \\ &= r + jx, \end{aligned}$$

where the resistance of the circuit is

$$r = \frac{e_1 i_1 + e_2 i_2}{i_1^2 + i_2^2} = \frac{e_1 i_1 + e_2 i_2}{I^2}$$

and the reactance of the circuit is

$$x = \frac{e_2 i_1 - e_1 i_2}{i_1^2 + i_2^2} = \frac{e_2 i_1 - e_1 i_2}{I^2}.$$

The admittance of the circuit is

$$\begin{aligned} y = \frac{I}{E} &= \frac{i_1 + ji_2}{e_1 + je_2} = \frac{i_1 + ji_2}{e_1 + je_2} \times \frac{e_1 - je_2}{e_1 - je_2} \\ &= \frac{e_1 i_1 + e_2 i_2}{e_1^2 + e_2^2} - j\frac{e_2 i_1 - e_1 i_2}{e_1^2 + e_2^2} \\ &= g - jb; \end{aligned}$$

the conductance is

$$g = \frac{e_1 i_1 + e_2 i_2}{e_1^2 + e_2^2} = \frac{e_1 i_1 + e_2 i_2}{E^2}$$

and the susceptance is

$$b = \frac{e_2 i_1 - e_1 i_2}{e_1^2 + e_2^2} = \frac{e_2 i_1 - e_1 i_2}{E^2}.$$

The power factor of the circuit is

$$\begin{aligned}\cos\phi &= \cos(\phi_1 - \phi_2) \\ &= \cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2 \\ &= \frac{e_1 i_1 + e_2 i_2}{EI}.\end{aligned}$$

The power consumed in the circuit is

$$\begin{aligned}P &= EI\cos\phi \\ &= EI\,\frac{e_1 i_1 + e_2 i_2}{EI} \\ &= e_1 i_1 + e_2 i_2, \qquad (196)\end{aligned}$$

and is the sum of the products of the components of the e.m.f. and current which are in phase. The products of the components of the e.m.f. and current which are in quadrature, namely, $e_1 i_2$ and $e_2 i_1$, do not represent power consumed.

The power may also be represented as

$$P = I^2 r = e_1 i_1 + e_2 i_2,$$

or

$$P = E^2 g = e_1 i_1 + e_2 i_2. \qquad (197)$$

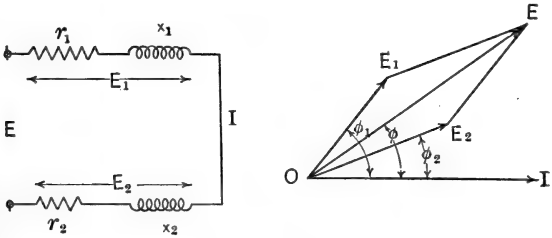


FIG. 105.

86. Examples in Rectangular Coordinates. (1) Find the current in the circuit in Fig. 105 in terms of the impressed e.m.f. and the constants of the circuit.

$$\begin{aligned}\underset{\cdot}{E}_1 &= \underset{\cdot}{I}\,(r_1 + jx_1). \\ \underset{\cdot}{E}_2 &= \underset{\cdot}{I}\,(r_2 + jx_2). \\ \underset{\cdot}{E} &= \underset{\cdot}{E}_1 + \underset{\cdot}{E}_2 = \underset{\cdot}{I}\,\{(r_1 + r_2) + j\,(x_1 + x_2)\}.\end{aligned}$$

The impedance of the circuit is

$$\dot{Z} = (r_1 + r_2) + j(x_1 + x_2)$$

and its absolute value is

$$Z = \sqrt{(r_1 + r_2)^2 + (x_1 + x_2)^2},$$

and the absolute value of the current is

$$I = \frac{E}{Z} = \frac{E}{\sqrt{(r_1 + r_2)^2 + (x_1 + x_2)^2}};$$

the power factor of the circuit is

$$\cos\phi = \frac{r_1 + r_2}{Z}.$$

The vector diagram is drawn taking the current as the axis.

(2) Solve the circuit in Fig. 106.

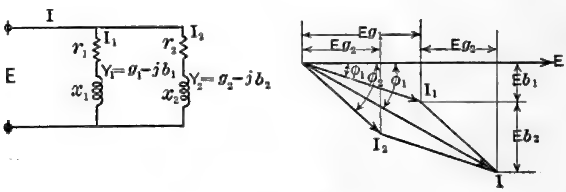


FIG. 106.

$$\dot{I}_1 = \frac{E}{r_1 + jx_1} = \dot{E}(g_1 - jb_1)$$

where $\quad g_1 = \dfrac{r_1}{r_1^2 + x_1^2} \quad$ and $\quad b_1 = \dfrac{x_1}{r_1^2 + x_1^2};$

$$\dot{I}_2 = \frac{E}{r_2 + jx_2} = \dot{E}(g_2 - jb_2)$$

the main current is

$$\dot{I} = \dot{I}_1 + \dot{I}_2 = \dot{E}\{(g_1 + g_2) - j(b_1 + b_2)\}$$

and its absolute value is

$$I = E\sqrt{(g_1 + g_2)^2 + (b_1 + b_2)^2}.$$

The admittance of the circuit is

$$Y = (g_1 + g_2) - j(b_1 + b_2)$$

and its absolute value is

$$Y = \sqrt{(g_1 + g_2)^2 + (b_1 + b_2)^2}.$$

The power factor of the circuit is

$$\cos \phi = \frac{g_1 + g_2}{Y}.$$

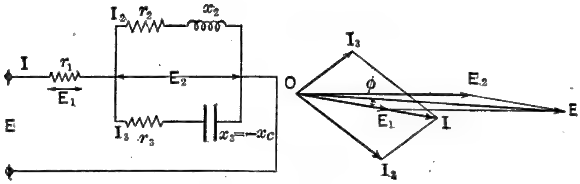


FIG. 107.

The vector diagram is drawn with the e.m.f. as axis.

(3) Solve the circuit in Fig. 107.

$$\dot{I}_2 = \frac{\dot{E}_2}{r_2 + jx_2} = \dot{E}_2 (g_2 - jb_2),$$

$$\dot{I}_3 = \frac{\dot{E}_2}{r_3 + jX_3} = \dot{E}_2 (g_3 - jb_3),$$

$$\dot{I} = \dot{I}_2 + \dot{I}_3 = \dot{E}_2\{(g_2 + g_3) - j(b_2 + b_3)\},$$

and therefore

$$\dot{E}_2 = \frac{\dot{I}}{(g_2 + g_3) - j(b_2 + b_3)} = \dot{I}(R + jX),$$

where

$$R = \frac{g_2 + g_3}{(g_2 + g_3)^2 + (b_2 + b_3)^2} \text{ and } X = \frac{b_2 + b_3}{(g_2 + g_3)^2 + (b_2 + b_3)^2},$$

$$\dot{E}_1 = \dot{I} r_1$$

and

$$\dot{E} = \dot{E}_1 + \dot{E}_2 = \dot{I}(r_1 + R + jX).$$

The impedance of the circuit is

$$\dot{Z} = r_1 + R + jX$$

and its absolute value is

$$Z = \sqrt{(r_1 + R)^2 + X^2}.$$

The absolute value of the current is

$$I = \frac{E}{Z}.$$

The power factor of the circuit is

$$\cos \phi = \frac{r_1 + R}{Z}.$$

The vector diagram is drawn with E_2 as the axis.

(4) If an e.m.f. $\dot{E} = 14 + j\,38$ is impressed on a circuit and a current $\dot{I} = 6 + j\,2$ flows, find the impedance of the circuit.

The impedance is

$$\dot{Z} = \frac{\dot{E}}{\dot{I}} = \frac{14 + j\,38}{6 + j\,2}$$

$$= \frac{14 + j\,38}{6 + j\,2} \times \frac{6 - j\,2}{6 - j\,2}$$

$$= \frac{160 + j\,200}{40} = 4 + j\,5;$$

the resistance of the circuit is 4 ohms and the inductive reactance is 5 ohms.

(5) In Fig. 108 a condensive reactance x_c is connected across the terminals of a receiver circuit of variable impedance $\dot{z} = r +$

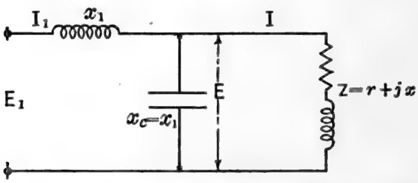


FIG. 108. Constant potential to constant current.

jx. If an inductive reactance $x_1 = x_c$ is connected in the supply lines and a constant e.m.f. E_1 is impressed on the terminals, show that the current in the receiver circuit is constant independent of the impedance and power factor.

$$\dot{E} = \text{e.m.f. at terminals of receiver,}$$

$$\dot{I} = \frac{\dot{E}}{\dot{z}} = \text{current in receiver,}$$

$$\dot{I}_c = \frac{\dot{E}}{-jx_c} = \text{current in condensive reactance,}$$

$$\dot{I}_1 = \dot{I} + \dot{I}_c = \text{current in the line,}$$

$$\dot{E}_1 = \dot{E} + j\dot{I}_1 x_1 = \text{e.m.f. impressed.}$$

Substituting the values above

$$\dot{E}_1 = \dot{E} + jx_1(\dot{I} + \dot{I}_c)$$
$$= \dot{E} + jx_1\left(\frac{\dot{E}}{\dot{z}} - \frac{\dot{E}}{jx_c}\right)$$
$$= \dot{E}\left(1 + j\frac{x_1}{\dot{z}} - \frac{jx_1}{jx_c}\right)$$
$$= j\frac{\dot{E}}{\dot{z}}x_1, \text{ since } x_c = x_1,$$
$$= j\dot{I}x_1,$$

or in absolute values

$$E_1 = Ix_1$$

and

$$I = \frac{E_1}{x_1}.$$

Since E_1 is constant I is constant independent of the impedance and the power factor of the receiver circuit. This circuit therefore transforms power from constant potential to constant current.

87. Kirchoff's Laws Applied to Alternating-current Circuits. Kirchoff's two laws enunciated in Art. 60 apply directly to alternating-current circuits when dealing with instantaneous values of e.m.f.'s and currents; they also apply to the effective values of e.m.f.'s and currents when combined in their proper phase relations. Thus the vector sum of all the e.m.f.'s around a closed circuit is zero if the e.m.f. consumed by resistance is considered as a counter e.m.f. in phase opposition to the current; and the vector sum of all currents at a distributing point is zero.

CHAPTER IV

DIRECT-CURRENT MACHINERY

88. The Direct-current Dynamo. A direct-current dynamo consists of an electric circuit, connected to a commutator and tapped by brushes, revolving in a magnetic field which is produced by stationary electric circuits.

Such a machine is illustrated in Fig. 109 and comprises the following parts:

1. Yoke 2. Pole pieces 3. Armature core	Magnetic circuit
4. Armature winding 5. Commutator	Revolving electric circuit.

6. Brushes and brush holders. Collecting apparatus.
7. Field winding. Stationary exciting circuit.

89. Yoke. The yoke serves mechanically as the frame of the machine and magnetically to carry the flux from pole to pole; it is usually made of cast iron but in machines where great weight is objectionable it is sometimes made of cast steel which has greater strength and permeability.

90. Pole Pieces. The pole pieces or pole cores are usually made of cast steel or sheet steel and are bolted to the yoke. For small machines the yoke and poles are sometimes cast in one piece. All solid poles must have laminated pole faces bolted to them in order to reduce the eddy current loss due to local variations of the magnetic density in the pole faces as the armature teeth move across them.

The pole cores carry the field windings of the machine. Solid poles are made circular and so have the greatest section for a given perimeter and require the smallest length of field copper. Laminated poles must, however, be made rectangular.

The section of the pole face is made much greater than that of the pole core in order to reduce the flux density in the air gap.

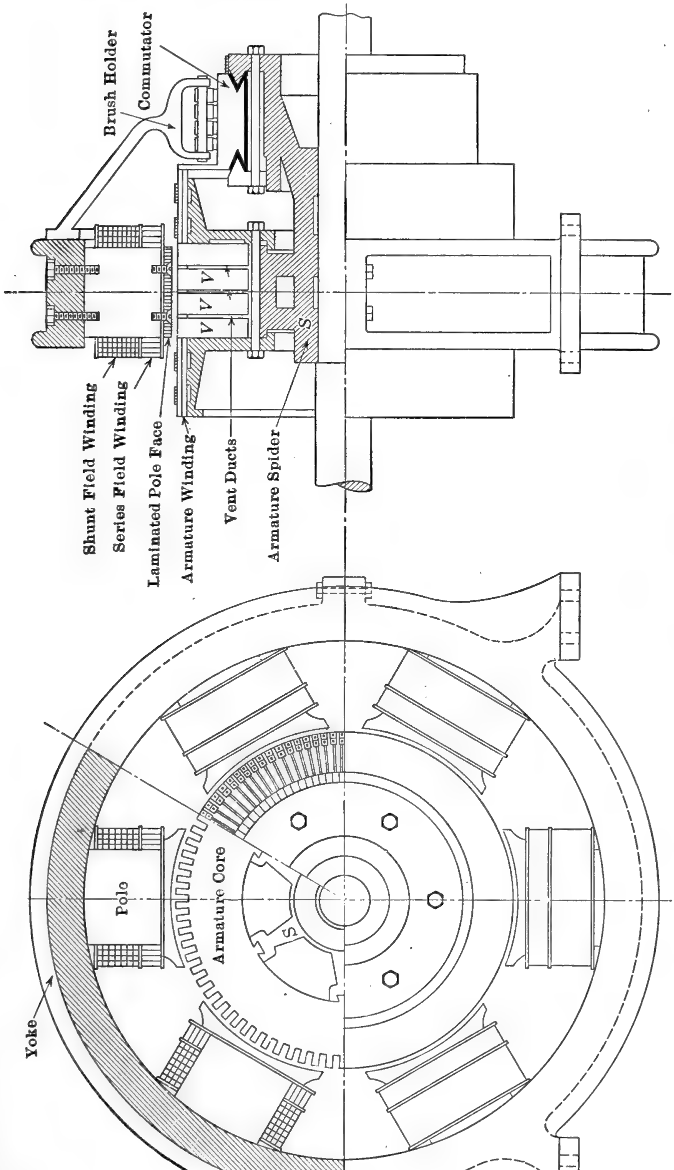


FIG. 109. Direct-current dynamo.

91. Armature Core. The armature core carries the rotating electric circuit in slots punched out on its periphery. It is built up of sheets of steel about 0.014 inch in thickness. Alternate sheets are coated with an insulating varnish to increase the resistance in the path of the induced eddy currents. Open spaces are left in the core, called vent ducts (*v.v.* Fig. 109), which allow air to circulate through the armature and carry off the heat generated due to the iron and copper losses. The number of vent ducts required depends on the length of the armature.

The armature punchings are carried on a spider *s* and are kept in place by heavy end plates which have projections on their outer edges to support the end connections of the armature coils.

92. Armature Winding. The armature winding is the seat of the generated electromotive force. It must be tapped at certain points by brushes, in order that the machine may supply power to an external receiver circuit. The winding consists of a number of coils of one or more turns, connected together to form a continuous winding; leads are run from their junctions to the commutator bars from which the current is collected by the brushes. The coils forming the winding must be so connected together that the e.m.f.'s generated in coils between brushes of opposite polarity will all act in the same direction.

The earliest type of armature winding was the ring winding,

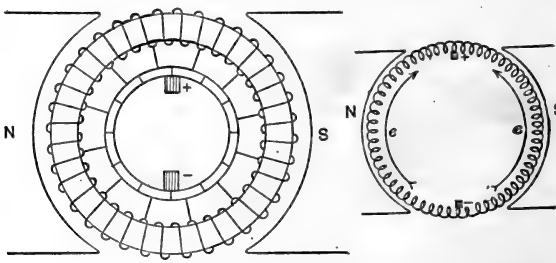


Fig. 110. Bipolar ring winding.

Figs. 110 and 111, but this has been replaced by the various forms of drum windings, a few of which are illustrated in Figs. 113 to 118.

93. Ring Windings. In the bipolar ring winding, Fig. 110, all the conductors on each half of the armature are connected in series between the brushes. When the brushes are placed on the neutral line, that is, in such a position that the coil being com-

mutated is not generating any e.m.f., the e.m.f.'s generated in all conductors under one pole will act in the same direction and will combine to give the terminal e.m.f. of the generator. The e.m.f.'s generated under the other pole will be equal in magnitude but will act in the opposite direction. Thus, there is no e.m.f. tending to cause current to circulate through the winding at no load and there are two paths in multiple for the current flowing through the armature.

The connection from one conductor to the next is run through inside the armature, where it cannot cut magnetic flux, and consequently one half of the winding is not effective in generating e.m.f. This extra wire increases the resistance of the armature and adds to the weight and cost of the machine. The ring winding has the further disadvantage, that it is very difficult to replace injured coils. On the other hand, the voltage between adjacent coils is so low that very little insulation is required between them. This type of winding is now obsolete.

Fig. 111 shows a six-pole ring winding with 36 coils connected to a commutator with 36 bars. This winding must be tapped at

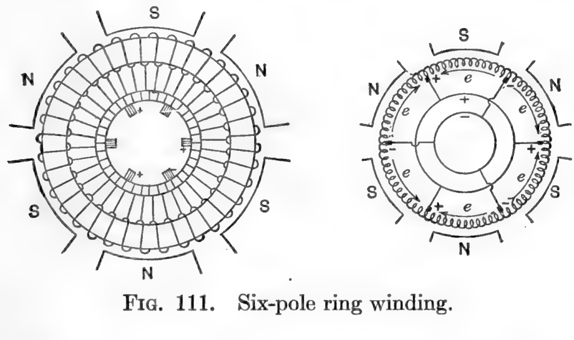


FIG. 111. Six-pole ring winding.

six equidistant points by brushes; there are six paths in parallel through the armature from positive to negative terminals and the voltage of the machine is that generated in one sixth of the winding.

94. Drum Winding. In drum-wound machines the whole of the armature winding is carried in slots on the outside of the armature core and both sides of any coil are effective in generating electromotive force. The single coils are of the shapes shown in Fig. 112 and may consist of one or more turns.

The conductors forming the two sides of a coil must be situated

in fields of opposite polarity in order that the electromotive forces generated in them may act in the same direction. One side of a coil is placed in the top of a slot and the other side in the bottom of a slot in a similar position under the next pole.

According to the way in which the end connections are brought out to the commutator bars and the coils are connected together, drum windings are divided into two classes, multiple or lap windings, as illustrated by coils *a* and *b* in Fig. 112 and the windings in Figs. 113 to 115, and series or two-circuit windings, as illustrated by the coils *c* and *d* in Fig. 112 and the windings in Figs. 116 and 117.

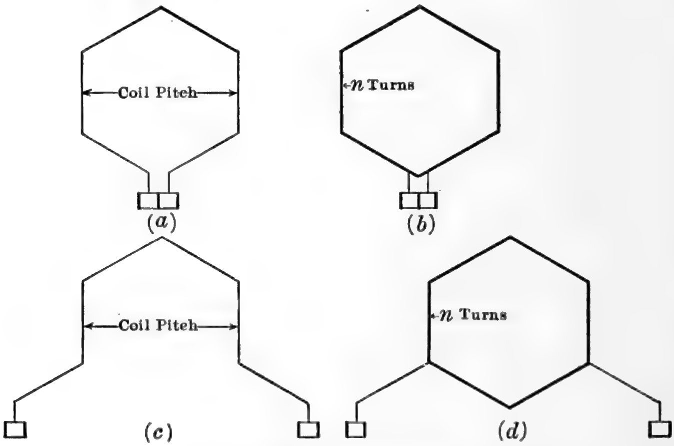
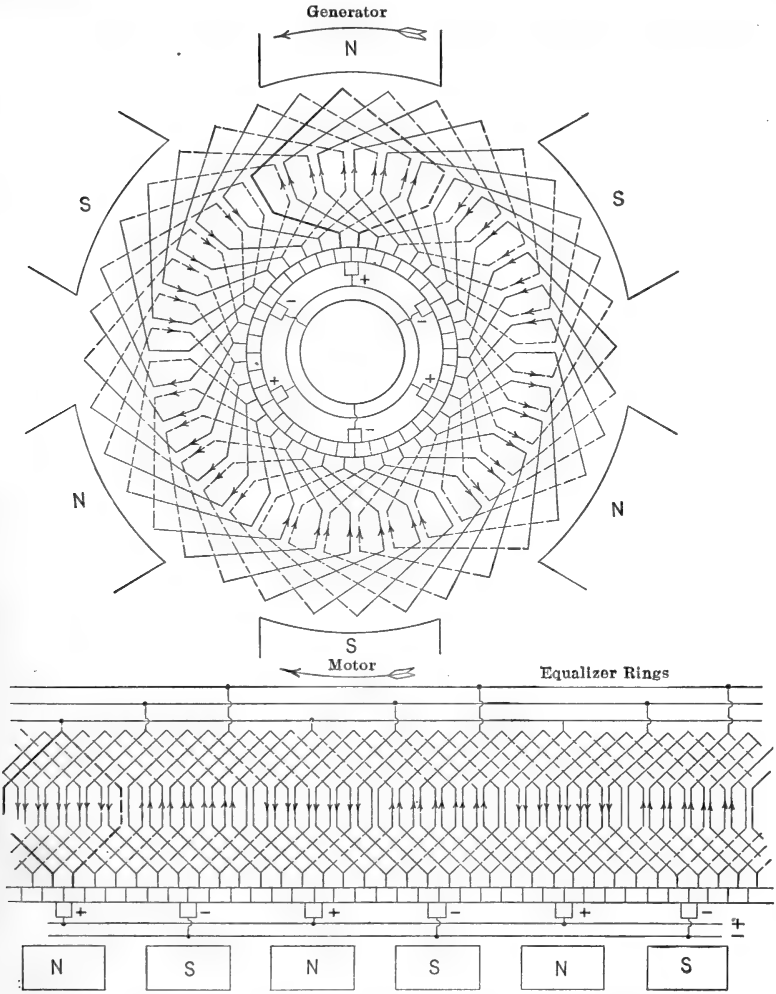


FIG. 112. Armature coils.

95. Multiple-drum Windings. In the multiple winding the two terminals of a coil are connected to adjacent commutator bars. Fig. 113 represents a multiple winding for a six-pole machine with 72 conductors and 36 slots. The sides of a coil are placed in slots 1 and 7 and the terminals are connected to bars 1 and 2. The same winding is shown in Fig. 114 and the directions of the currents are shown by arrowheads. The brushes are placed on the no-load neutral points and therefore directly under the centres of the poles and as many sets of brushes are required as there are poles.

Tracing through the winding from a positive to a negative brush only one sixth of the conductors are taken and there are therefore

six paths in multiple through the armature winding from the positive to the negative terminal of the machine and each conductor carries only one sixth of the current passing through the armature.



FIGS. 113 and 114. Six-pole multiple drum winding.

The number of paths is equal to the number of poles as in the ring winding.

Generally in multiple windings the coil pitch is almost equal to the pole pitch and the windings are called full-pitch windings.

When the coil pitch is less by one or more teeth than the pole pitch the winding is called a fractional pitch or short-chord winding.

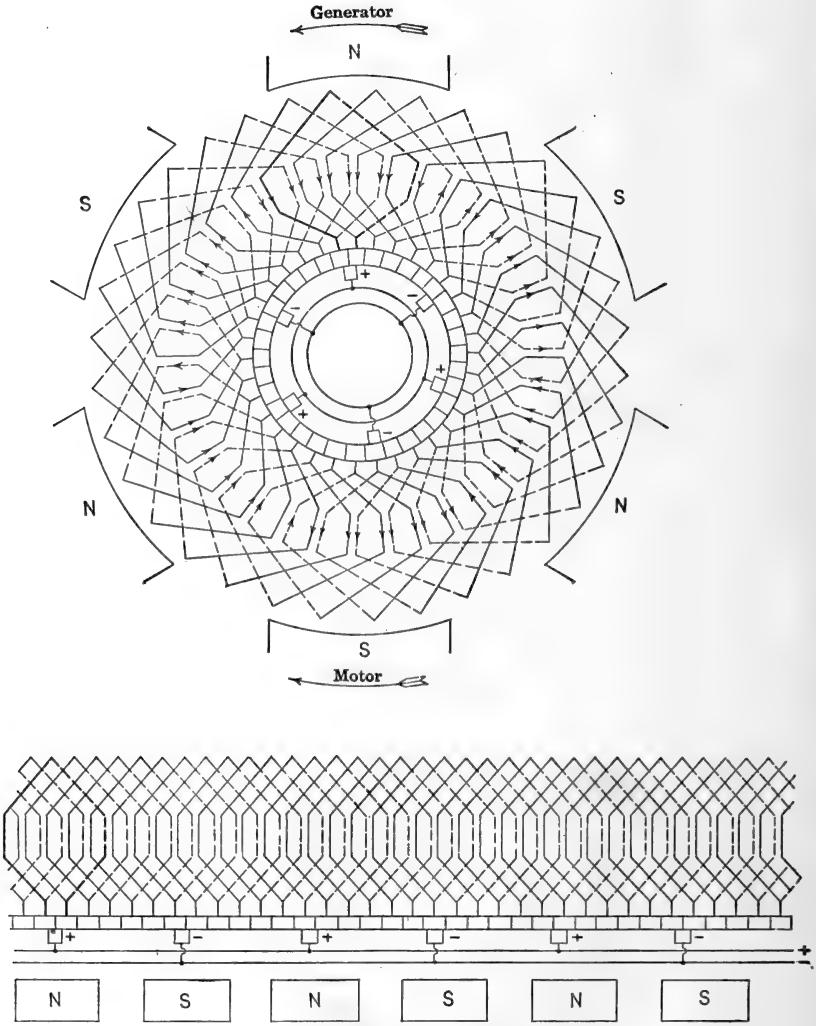


Fig. 115. Six-pole multiple-drum winding. Fractional pitch.

Fig. 115 shows a fractional pitch, multiple-drum winding. Fractional-pitch windings have shorter end connections than full-pitch windings and have a smaller inductive e.m.f. generated during

commutation since the two coils in one slot are not commutated at the same time and thus the inductive flux is that due to half the ampere turns acting in the case of a full-pitch winding.

96. Equalizer Rings. In multiple-wound machines, if there is any irregularity in spacing the brushes or if the air gaps under all the poles are not of the same depth, the e.m.f.'s generated in the different sections of the winding will not be equal and the unbalanced e.m.f. will tend to cause current to circulate through the windings even when the machine is not carrying any load. To reduce the circulating currents similar points under the different pairs of poles which should normally be at the same potential are joined together by heavy copper connections called equalizer rings (Fig. 114) and these prevent the current from circulating through the windings.

97. Series-drum Windings. In the series winding the terminals of a coil are connected to two commutator bars approximately twice the pole pitch apart. Fig. 116 represents a series or two-circuit winding for a six-pole machine with 44 conductors and 22 slots. One side of a coil is placed in the top of slot 1 and the other side in the bottom of slot 5 and the terminals of the coil are connected to commutator bars 1 and 8.

Tracing out the winding from the positive brush B_1 to the negative brush B_2 one half of the armature conductors are taken in. There are therefore but two circuits in multiple between terminals independent of the number of poles and the winding is called a two-circuit or series winding.

Only two sets of brushes are required to collect the current but when the current is large it is usual to employ other sets of brushes as shown at B_3, B_4, B_5 and B_6 and as many sets of brushes as there are poles may be used.

Series windings are used in small high-voltage machines or where it is desirable to use only two sets of brushes, as in small railway motors; but in large multipolar machines with many sets of brushes the current does not divide equally between brushes of the same polarity and commutation is unsatisfactory.

The number of coils in a series winding must be one more or one less than a multiple of the number of pairs of poles, or

$$N = C\frac{p}{2} \pm 1,$$

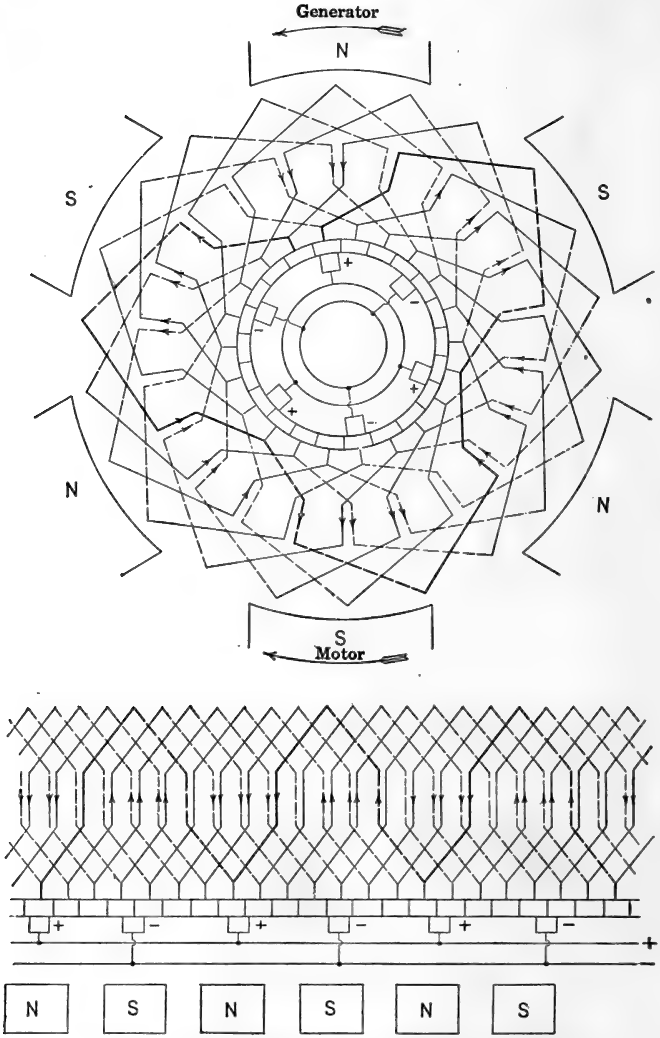


FIG. 116. Six-pole series drum winding, retrogressive.

where N = number of armature coils,

$p/2$ = numbers of pairs of poles,

and C = a constant whole number.

Each coil may have any number of turns.

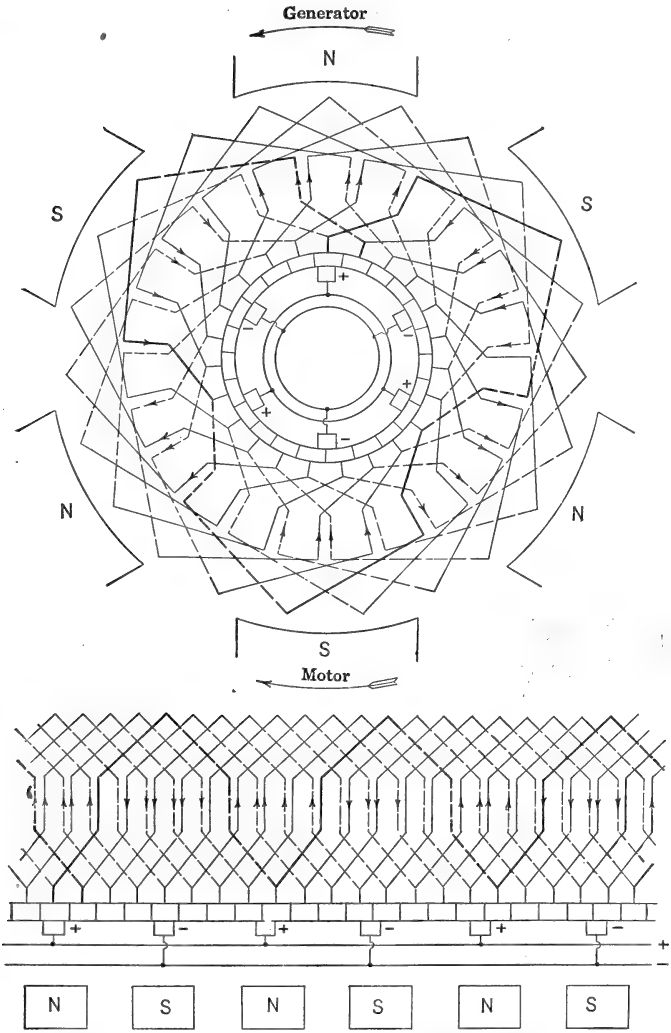


FIG. 117. Six-pole series-drum winding, progressive.

If

$$N = C\frac{p}{2} - 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (198)$$

the winding starting from bar 1 goes once around the armature and is connected to bar 2. It is therefore called a progressive winding. (Fig. 117.)

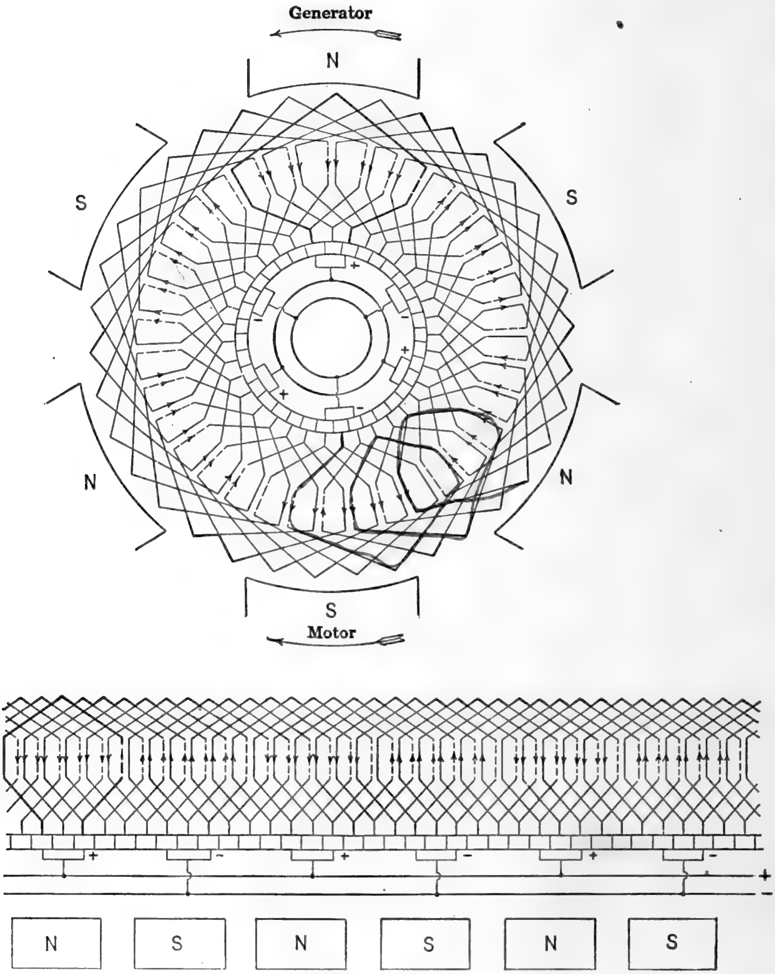


FIG. 118. Six-pole multiple-drum winding, duplex, doubly reëntrant.

If

$$N = C\frac{p}{2} + 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (199)$$

the winding starting from bar 1 and going once around the armature is connected to the bar before 1 and it is called a retrogressive winding. (Fig. 116.)

98. Double Windings. If space is left between adjacent coils of a multiple winding, a second winding may be placed on the same core. The second winding may be entirely separate from the first, that is, each of the two windings closed upon itself; or after passing through the first winding the circuit enters the second and after passing through the second reënters the first. In the first case the winding is duplex doubly reëntrant and in the second case duplex singly reëntrant. Duplex multiple windings have twice as many circuits in multiple between terminals as there are poles. Such windings are suitable for large low-voltage machines used in electrolytic work. The brushes must be wide enough to collect current from both sections of the winding at the same time. Fig. 118 shows a duplex doubly reëntrant winding for a six-pole machine with 72 conductors and 36 slots.

Similarly the series winding may be made double by placing a second winding in alternate slots and connecting it to alternate commutator bars. The second winding is in multiple with the first and there are four paths in multiple between terminals.

99. Commutator. The commutator is one of the most important parts of a direct-current machine. It consists of a number

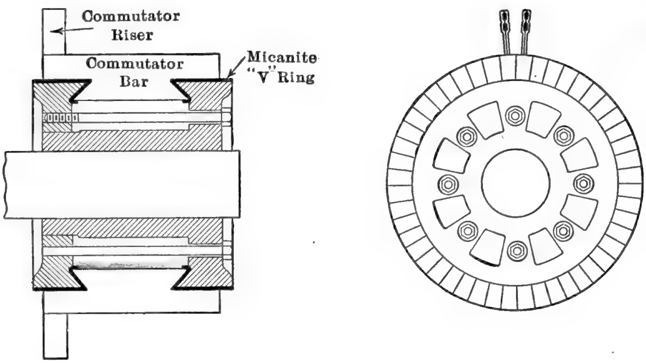


FIG. 119. Commutator.

of bars of hard-drawn copper, insulated from one another by thin sheets of mica or other insulating material, and built up into the form of a cylinder. (Fig. 119.) The bars are held together by a cast-iron spider from which they are insulated by micanite "V" rings. The terminals of the coils forming the armature winding are connected to the bars either directly by soldering them into slots in

the bars or by means of vertical connectors called commutator risers. In order that the brushes, which collect the current from the commutator, may run smoothly without vibrating or chattering the commutator surface must be perfectly round and smooth.

The function of the commutator is illustrated in Fig. 120. The current from the machine is I amperes and the current in each conductor is $I_c = \frac{I}{2}$ amperes. During the time taken for the brush to move across the insulation between bars 2 and 3 the current in coil c must change from I_c in one direction to I_c in the opposite

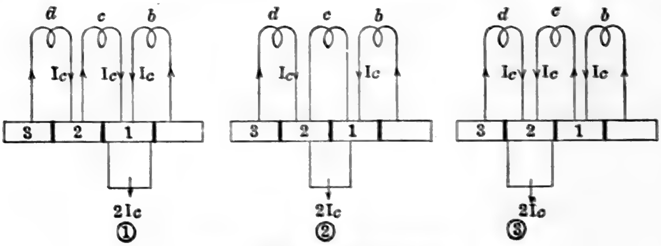


Fig. 120. Commutation.

direction. This reversal of the current is called commutation and to be satisfactory it must be effected without sparking. The brushes are shown placed on the neutral line and the coils short circuited are not cutting any flux and therefore have no e.m.f. generated in them due to rotation. If the short-circuited coil had no inductance the current would reverse completely due to the contact resistance between the brush and the commutator.

In (1) the current in coil c is I_c; in (2) it is zero since current I_c from bar 1 goes through one half of the brush-contact area and current I_c from bar 2 goes through the other half, and the drop of voltage on both sides is the same and therefore there is no voltage available to drive the current through the resistance of the coil. Between (1) and (2) the resistance from bar 1 to the brush is greater than the resistance from bar 2 to the brush and so part of the current from 2 flows through the coil c. Between (2) and (3) the resistance from bar 2 to the brush is less than from bar 1 and part of the current from 1 flows through coil c. In (3) the current in c is I_c but in the opposite direction from that in (1) and commutation is complete.

The self-inductance L of the coil opposes any change of current by generating a back e.m.f. $L\frac{di}{dt}$ volts; when the current is large and the time of commutation short this back e.m.f. is large and the current will not be reversed when the brush breaks contact with bar 1 and sparking will occur.

To counteract the effect of self-inductance the brushes in a generator are moved ahead of the neutral in the direction of rotation and back in a motor. The short-circuited coil is then in a field which generates in it an e.m.f. due to rotation which opposes the back e.m.f. of self-inductance, or, as it is usually called, "the reactance voltage of the coil," and assists commutation. The problem of commutation is discussed fully in Art. 110.

100. Brushes and Brush Holders. The brushes collect the current from the moving commutator and from them it passes to the receiver circuit.

Brushes were at first made of copper because it had a low resistance and large current-carrying capacity but commutation of large currents was not satisfactory. Carbon brushes were then introduced and commutation was greatly improved due to the action of the high-resistance contact film between the brush and commutator. A much better contact surface was also obtained and the wear on the commutator was reduced. But since carbon will only carry about 40 amperes per square inch while copper will carry 300 amperes per square inch a much larger brush area is required and a larger commutator.

In order to maintain a good contact between the brush and commutator a spring is used exerting on the brush a pressure of about $2\frac{1}{2}$ pounds per square inch of contact area.

The brush holders are made of brass and carry part of the current but leads are connected directly from the brushes to the main leads of the machine to prevent any drop of voltage which might occur due to poor contact between brushes and holders.

101. Field Windings. The field winding is a stationary electric circuit consisting of one or more coils of wire placed on each of the field poles. They are supplied with current and provide the magnetomotive force necessary to drive the magnetic flux through the machine. The method used in calculating the number of ampere turns required to produce the flux in a machine is worked out in Art. 107.

According to the manner of exciting the magnet fields, direct-current machines are divided into magneto machines in which the flux is produced by permanent magnets; separately excited ma-

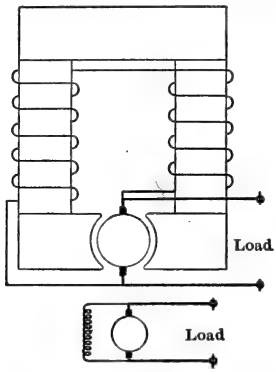


FIG. 121. Shunt or self excitation.

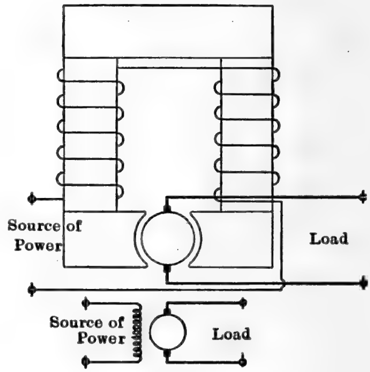


FIG. 122. Separate excitation.

chines (Fig. 121) in which the flux is produced by a winding supplied with current from some source outside the machine; shunt machines in which the field winding is connected across the arma-

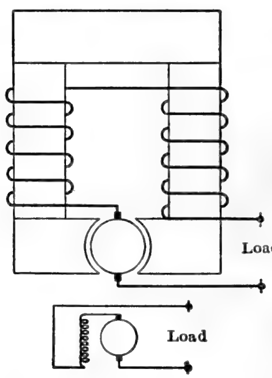
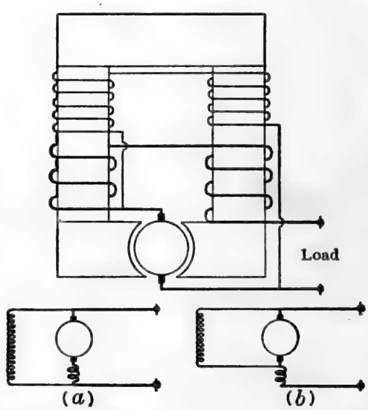


FIG. 123. Series excitation.



(*a*) Long Shunt (*b*) Short Shunt

FIG. 124. Compound excitation.

ture terminals and receives a small current at the full-machine voltage (Fig. 122); series machines in which the field winding is connected in series with the armature and carries the full arma-

ture current (Fig. 123); and compound machines in which the field has both a shunt and a series winding (Fig. 124).

According to the number of poles machines are divided into bipolar and multipolar machines but the bipolar type is only adapted for small sizes.

102. Direction of Rotation of Generators and Motors. Fig. 125 represents either a generator or motor. The directions of the currents in the armature are shown by the dots and crosses and the directions of rotation by arrows.

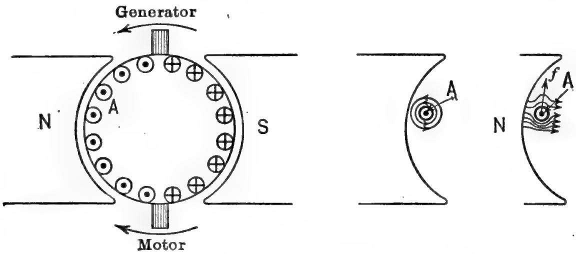


FIG. 125. Direction of rotation.

With currents as shown, the direction of rotation of a generator is counter-clockwise and of a motor is clockwise.

The direction of the e.m.f. of a generator is the direction of the current but the generated e.m.f. in a motor is opposed to the current.

These results are obtained by examining the fields produced by the armature currents.

Take for example the conductor *A*. Its field combines with the main field and produces a strong field below the conductor and a weak field above it. There is therefore a force *f* acting on the conductor tending to move it up. This is the force that must be overcome by the engine driving the generator in order to develop electric power and therefore the rotation of a generator is against this force and is counter-clockwise.

In the case of the motor, the force *f* on the conductor is the mechanical force developed and the rotation is in the direction of this force and is clockwise.

To reverse the direction of rotation of a motor it is necessary to reverse either the armature current or the field current but not both.

103. Generation of Electromotive Force. The electromotive force generated in the armature of a direct-current generator or motor is

$$\mathcal{E} = Zn\Phi \frac{p}{p_1} 10^{-8} \text{ volts,}$$

where Z = number of conductors on the armature,
n = speed of armature in r.p.s.,
Φ = flux crossing the air gap from one pole,
p = number of poles,
and p_1 = number of paths in multiple between terminals.

In one second each conductor cuts $n\Phi p$ lines of force and thus the average e.m.f. generated in each of the Z conductors is

$$e = n\Phi p \, 10^{-8} \text{ volts.}$$

Between the terminals there are $\frac{Z}{p_1}$ conductors connected in series and therefore the e.m.f. between terminals is

$$\mathcal{E} = Zn\Phi \frac{p}{p_1} 10^{-8} \text{ volts.} \quad \quad (200)$$

This is the electromotive force equation of a direct-current generator. In a motor it is called a back electromotive force since it opposes the impressed e.m.f. and therefore the current in the motor armature.

This equation may be written

$$\mathcal{E} = Kn\Phi, \quad \quad (201)$$

where $K = Z\frac{p}{p_1} 10^{-8}$ is a constant of the machine.

The electromotive force is therefore directly proportional to the speed and to the flux crossing the air gap.

104. Effect of Moving the Brushes. The equation

$$\mathcal{E} = Zn\Phi \frac{p}{p_1} 10^{-8}$$

only holds if the brushes are on the no-load neutral points. When the brushes are moved ahead of the neutral points or

behind them the e.m.f. between terminals is decreased. This may be seen by reference to Fig. 126. With the brushes on the neutral points the e.m.f.'s generated in all the conductors in series between terminals act in the same direction and combine to give the maximum e.m.f. When the brushes are moved ahead the e.m.f. between terminals is only that generated in conductors *a–b* or *d–c* since the resultant of the e.m.f.'s generated in conductors *a–d* and *b–c* is zero. Thus advancing the brushes corresponds to a decrease in the number of armature conductors.

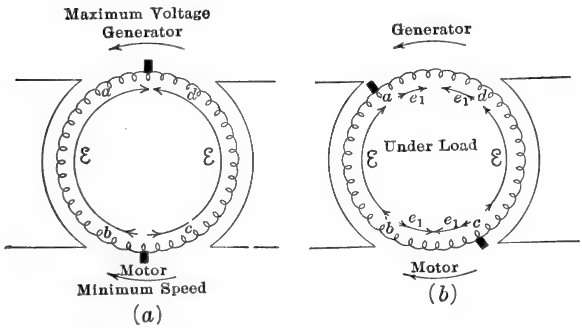


FIG. 126. Effect of moving the brushes.

105. Saturation Curve. The flux Φ crossing the air gap can be varied by varying the current in the field winding. This is accomplished by connecting a field rheostat in series with the field winding. When the resistance is increased the current decreases and the flux decreases as shown in Fig. 127, which is the saturation curve of a machine plotted with flux per pole on a base of field current.

At first the flux increases almost directly as the current, while the iron parts of the circuit are unsaturated, but as the flux density increases the magnetic circuit becomes saturated and a greater increase of current is required to produce a given increase of flux than on the lower part of the curve.

Since the e.m.f. generated varies directly with the flux, by multiplying the ordinates of the curve (1) in Fig. 127 by the constant *Kn*, curve (2) is obtained of the same shape as before giving the generated e.m.f. or terminal e.m.f. at no load on a base of field current.

This curve is the no-load saturation curve or magnetization curve of the machine.

Shunt-excited machines are operated at a point slightly above the knee of the saturation curve to secure stability, that is, in order to prevent large changes of e.m.f. being caused by slight changes of field current.

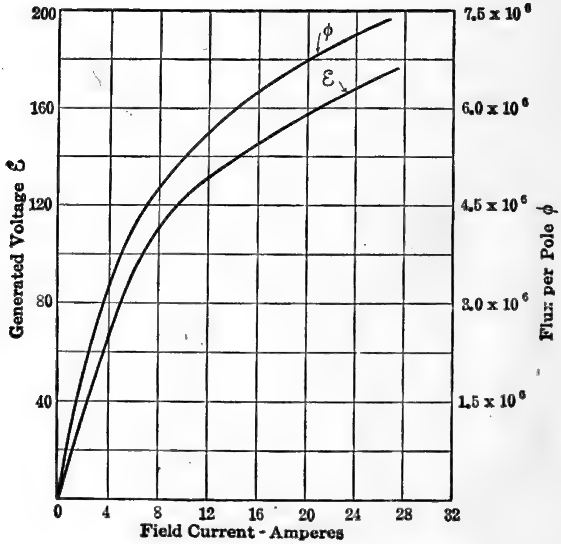


FIG. 127. No-load saturation curve.

106. Magnetic Leakage. Since there is no material through which magnetic flux cannot pass, it is not possible to confine all the flux produced in a generator to the magnetic circuit. In practice the main circuit is made of so low a reluctance that only a small portion of the flux leaves it.

Fig. 128 shows the leakage flux about the magnetic circuit of a bipolar generator and Fig. 129 shows it in the case of a multipolar generator.

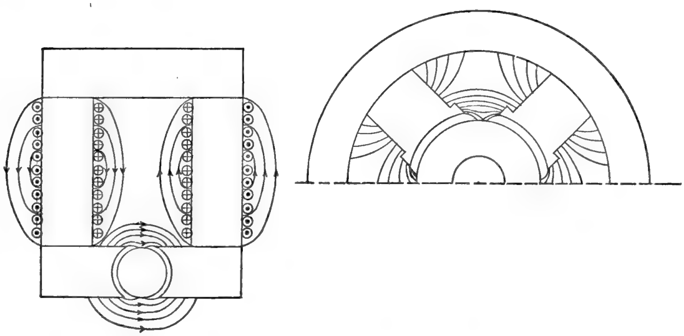
The principal part of the leakage occurs between the pole tips because the m.m.f. consumed between these points is from 60 to 80 per cent of the total m.m.f.; it includes the m.m.f. required to drive the flux across the two gaps and through the teeth and armature core. As a result the flux passing through the field poles is greater than the flux crossing the gap into the armature by an

amount depending both on the mechanical construction of the machine and on the load.

The dispersion coefficient or leakage factor is the ratio of the flux through the field poles to the flux crossing the gap into the armature, that is, the ratio of the total flux to the useful flux; thus the leakage factor is

$$\nu = \frac{\phi \text{ pole}}{\phi \text{ gap}}.$$

ν varies from about 1.1 to 1.5 depending on the construction of the machine but usually has a value of from 1.15 to 1.2 at no load.



FIGS. 128 and 129. Leakage flux.

Under load the armature also exerts a m.m.f. which in part is demagnetizing and opposes the field m.m.f. and in part is cross magnetizing and increases the reluctance of the magnetic circuit. (See Art. 109.) Thus under load a greater proportion of the total m.m.f. of the field is required for the air gaps, teeth and armature than at no load and therefore the leakage flux is increased and at the same time the main flux is decreased. The leakage factor is therefore greater under load than at no load.

107. Determination of the No-load Saturation Curve of a Dynamo. Fig. 130 shows the dimensions of the magnetic circuit of a 6-pole, 150-kilowatt, 150-volt, 280 r.p.m. generator. The armature winding has 512 conductors and is multiple wound.

The voltage generated in the armature between brushes is

$$\mathcal{E} = Zn\Phi \frac{p}{p_1} 10^{-8} \text{ volts},$$

where n is the speed in r.p.s. $= \frac{280}{60}$,
Z is the number of armature conductors $= 512$,
Φ_g is the flux crossing the air gap,
p is the number of poles $= 6$,
p_1 is the number of paths in parallel through the armature winding $= 6$,
and $\mathcal{E}$ is the terminal voltage at no load;

therefore,

$$\mathcal{E} = 512 \times \frac{280}{60} \times \frac{6}{6}\, 10^{-8}\, \Phi_g,$$

or

$$\Phi_g = 42{,}000\ \mathcal{E}.$$

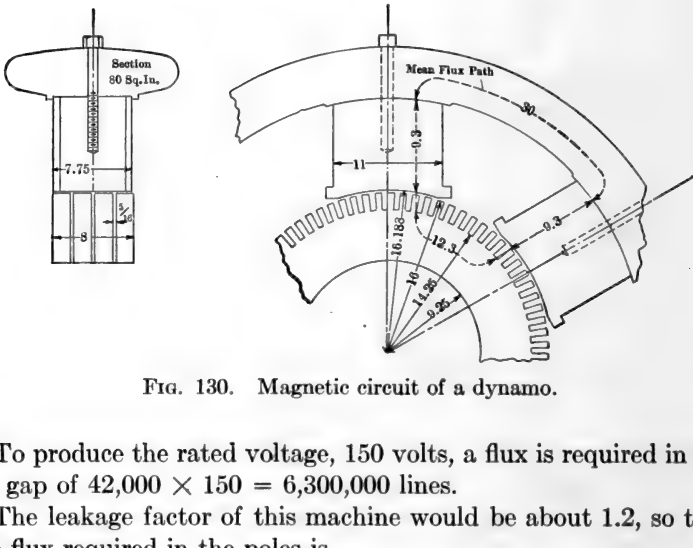


FIG. 130. Magnetic circuit of a dynamo.

To produce the rated voltage, 150 volts, a flux is required in the air gap of $42{,}000 \times 150 = 6{,}300{,}000$ lines.

The leakage factor of this machine would be about 1.2, so that the flux required in the poles is

$$\Phi_p = 1.2\,\Phi_g = 1.2 \times 6{,}300{,}000 = 7{,}560{,}000 \text{ lines.}$$

The following table (Fig. 131) gives the values Φ_p and Φ_g for values of terminal voltage from 0 to 175 volts, assuming that the leakage factor is constant.

The next step is to determine the lengths and sections of the various parts of the magnetic circuit and the ampere turns required per pair of poles to drive the flux through them.

The yoke is made of cast iron and has a section of 80 sq. ins.; the length of the magnetic path through it is taken as the estimated

length of the mean flux line through it and can best be obtained from the drawing; it is in this machine 30 ins.

Voltage	Flux in the pole, ϕ_p	Flux in the gap, ϕ_g
0	0	0
25	1,260,000	1,050,000
50	2,520,000	2,100,000
75	3,780,000	3,150,000
100	5,040,000	4,200,000
125	6,300,000	5,250,000
150	7,560,000	6,300,000
175	8,820,000	7,350,000

FIG. 131.

The flux passing through the yoke is one half of that in the pole and for a terminal voltage of 150 volts, it is

$$\Phi_y = \frac{\Phi_p}{2} = \frac{7,560,000}{2} = 3,780,000;$$

the flux density in the yoke is

$$B_y = \frac{\Phi_y}{A_y} = \frac{3,780,000}{80} = 47,250 \text{ lines per square inch.}$$

From the permeability curve for cast iron (Fig. 59) it is found that 100 ampere turns are required to drive this flux density through one inch length of the yoke, therefore the number of ampere turns required per pair of poles for the yoke is $30 \times 100 = 3000$.

The field poles are made of sheet steel and are of rectangular section, 11 ins. wide by 7.75 ins. deep along the shaft; the section is $11 \times 7.75 = 85.25$ square inches; the length of the path through one pole is 9.3 ins. and per pair of poles is 18.6 ins.

The flux density in the pole for 150 volts is $\frac{7,560,000}{85.25} = 88,800$ lines per square inch; from the curve for sheet steel (Fig. 59) it is found that 28 ampere turns per inch are required and $18.6 \times 28 = 520$ ampere turns per pair of poles.

The increased section of the pole face is neglected.

The air gap is the most important section in the magnetic circuit because the largest part of the field magnetomotive force is

consumed in driving the flux across it, and its section and length must be calculated very carefully.

The section of the air gap can be taken as the area of the pole face; from the drawing it is found to be 100 sq. ins.

If the length of the air gap is taken as the radial length from pole face to armature a large error will result because the average length of the lines crossing the gap from pole to armature is greater than the radial length as seen in Fig. 132.

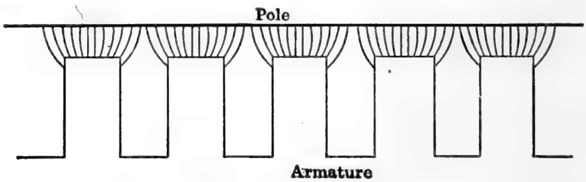


FIG. 132. Flux in the air gap.

The radial length must be multiplied by a constant greater than unity in order to give the correct length. Carter has derived values for this constant depending on the ratios of tooth width to slot width and slot width to gap length. For machines of ordinary design the constant ranges from 1.1 to 1.2 and a value of 1.17 has been taken in the present case. Therefore the corrected length of the air gap is $0.188 \times 1.17 = 0.2014$ in. under each pole and for a pair of poles is 0.4028 in.

The flux crossing the gap corresponding to 150 volts is 6,300,000 lines and the flux density is $\frac{6,300,000}{100} = 63,000$ lines per square inch.

The number of ampere turns required to drive a flux density of B lines per square inch through a length of one inch in air is found as follows:

If $\mathfrak{B}$ = lines per square centimeter, l cm. = the length of the path and nI = the ampere turns required, then

$$\mathfrak{B} = \frac{0.4\,\pi nI}{l}\mu,$$

but $$\mathfrak{B} = \frac{B}{(2.54)^2}, \quad l = 1 \text{ in.} = 2.54 \text{ cm. and } \mu = 1;$$

therefore,

$$\frac{B}{(2.54)^2} = \frac{0.4\,\pi nI}{2.54}$$

and the number of ampere turns required is

$$nI = \frac{B}{(2.54)^2} \times \frac{2.54}{0.4 \times 3.14} = 0.3133\,B. \quad . \quad . \quad (202)$$

Thus the ampere turns required per pair of poles for a density of 63,000 lines per square inch for the two air gaps is $0.4028 \times 0.3133 \times 63{,}000 = 7968$.

The teeth form the next section; they are projections of the armature core and are made of sheet steel.

The length of the path through the teeth per pair of poles is twice the depth of a slot $= 2 \times 1.75 = 3.5$ ins.

The section of the path through the teeth is taken as the mean iron section of one tooth multiplied by the number of teeth under one pole; the mean width of a tooth is 0.62 in.; the over-all length of the armature core is 8 ins. but from this must be subtracted 3 vent spaces $\frac{5}{16}$ in. in width and an allowance of 10 per cent must be made as a stacking factor, since the armature is built up of thin sheets insulated with varnish; thus, the length of iron in the tooth is $(8 - 3 \times \frac{5}{16}) \times 0.9$; the number of teeth under one pole allowing for fringing is 11; therefore the iron section of the path through the teeth is

$$11 \times 0.62 \times (8 - 3 \times \tfrac{5}{16}) \times 0.9 = 43.4 \text{ sq. ins.}$$

The flux passing through the teeth is the same as that through the air gap $= 6{,}300{,}000$ lines, and the flux density in the teeth is $\frac{60{,}300{,}000}{43.4} = 145{,}200$ lines per square inch; this requires 1500 ampere turns per inch length and a total of $1500 \times 3.5 = 5250$ ampere turns per pair of poles.

If the flux density in the teeth is above 100,000 lines per square inch it is necessary to take account of the fact that the path through the teeth is paralleled by an air path consisting of the slots, the vent ducts and the insulation between punchings; this path has usually a larger section than the path through the teeth and consequently at high densities, where the permeability of the iron is low, a considerable amount of the flux will follow this path.

In this case the area of the parallel air path is 58 sq. ins. and from Fig. 59 the permeability of the iron for a flux density of

145,200 lines per square inch is found to be 30; the permeance of the path through the teeth is

$$\mathcal{P}_1 = \frac{A_1\mu_1}{l_1} = \frac{43.4 \times (2.54)^2 \times 30}{3.5 \times 2.54} = 942,$$

and the permeance of the air path is

$$\mathcal{P}_2 = \frac{A_2}{l_2} = \frac{58 \times (2.54)^2}{3.5 \times 2.54} = 42;$$

the flux of 6,300,000 crossing the gap divides between the two paths in proportion to their permeances, and therefore the actual flux passing through the teeth is

$$6{,}300{,}000 \times \frac{\mathcal{P}_1}{\mathcal{P}_1 + \mathcal{P}_2} = 6{,}300{,}000 \times \frac{942}{942 + 42} = 6{,}040{,}000;$$

and the corrected density in the teeth is $\dfrac{6{,}040{,}000}{43.4} = 139{,}300$; the number of ampere turns per inch corresponding to this is 1200 and the number per pair of poles is $1200 \times 3.5 = 4200$. The error introduced in this case by considering that the teeth carried the whole flux would be $5250 - 4200 = 1050$ ampere turns which is an error of 25 per cent.

The last section is the armature core below the slots; its section is the product of the net length of iron in the armature by the depth of iron below the slots; it is $(8 - 3 \times \frac{5}{16}) \times 0.9 \times 5 = 31.5$ sq. ins.; the length of the path through the armature is estimated from the drawing to be 12.3 ins.

The flux carried by the armature section is

$$\frac{\Phi_g}{2} = \frac{6{,}300{,}000}{2} = 3{,}150{,}000 \text{ lines},$$

and the flux density is

$$\frac{3{,}150{,}000}{31.5} = 100{,}000 \text{ lines per square inch.}$$

The number of ampere turns per inch required is 75 and the number per pair of poles is $12.3 \times 75 = 860$.

The results of the calculations above are tabulated in Fig. 133; the total number of ampere turns required per pair of poles to produce a flux across the air gap of 6,300,000 lines and a voltage of 150 volts is 16,548.

Part of machine	Material	Magnetic length, inches	Section, sq. in.	Flux	Flux density, lines per sq. in.	Ampere turns per inch	Ampere turns per pair of poles
Yoke.......	Cast iron...	30.0	80.0	3.78×10^6	47,250	100	3000
Poles (2)...	Sheet steel.	18.6	85.25	7.56×10^6	88,800	28	520
Gaps (2)....	Air.........	0.403	100.0	6.30×10^6	63,000	$63{,}000\times0.3133$	7968
Teeth (2)...	Sheet steel..	3.5	43.4	6.04×10^6	139,300	1200	4200
Armature..	Sheet steel..	12.3	31.5	3.15×10^6	100,000	75	860

Total ampere turns per pair of poles = 16,548.

FIG. 133.

Similar calculations have been made for voltages from 25 to 175 volts and the results are tabulated in Fig. 134.

The curve in Fig. 135 is plotted with the voltages as ordinates and the ampere turns per pair of poles as abscissæ; it is the saturation or magnetization curve of the machine when running without load.

Volts	Ampere turns for yoke	Ampere turns for poles	Ampere turns for air gaps	Ampere turns for teeth	Ampere turns for armature	Ampere turns per pair of poles
25	90	23	1328	7	18	1,466
50	210	52	2656	18	37	2,973
75	450	84	3984	42	67	4,627
100	990	139	5312	185	123	6,749
125	2100	260	6640	1275	258	10,483
150	3000	520	7968	4200	860	16,548
175	5400	2046	9296	7430	2760	26,932

FIG. 134.

108. Building up of E. M. F. in a Self-excited Generator. In a self-excited generator at rest there is no flux crossing the gap except the residual magnetism. When the armature is rotated only a small e.m.f. is generated in it and a very small current is produced in the field winding. If the magnetomotive force of this current is in the direction of the residual magnetism, it will increase the flux and the e.m.f. will increase and gradually build up to its full value.

If, however, the current opposes the residual magnetism, it will cause it to decrease and the e.m.f. will not build up until the field winding is reversed.

If there is no residual magnetism, the e.m.f. cannot build up until power is supplied to the winding from some outside source to start the flux.

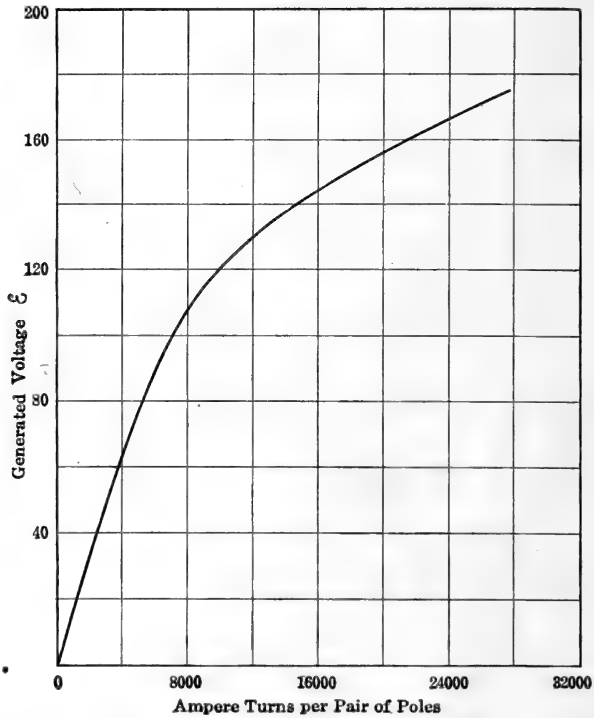


FIG. 135. No-load saturation curve.

109. Distribution of Magnetic Flux. Fig. 136 shows approximately the distribution of flux in a two-pole machine with the fields excited but without current in the armature winding. Curve 1 shows the m.m.f. acting at each point of the armature circumference; under the north pole it is positive and has a constant value; under the south pole it is negative but of the same magnitude; beyond the pole tips its value may be represented by the straight line which passes through zero midway between the poles. The m.m.f. is expressed in ampere turns and is denoted by $\mathcal{M}_f$.

The flux density produced by this m.m.f. is at every point directly proportional to the m.m.f. and is inversely proportional to the reluctance of the path. It is of constant value over the pole

face if the air gap is uniform but falls off rapidly beyond the pole tips due to the increased reluctance of the air path and to the decrease in the m.m.f. acting. Midway between the poles it is zero. It is represented by B and its values are plotted in curve 2. The total flux entering the armature is represented by the area under curve 2 and this is the value of Φ which appears in the e.m.f. equation.

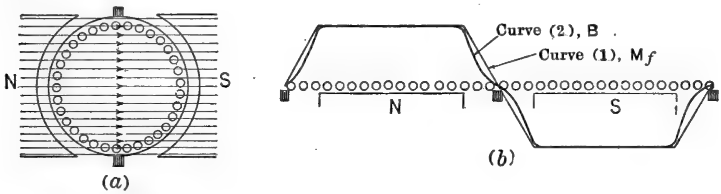


FIG. 136. Distribution of flux and m.m.f. at no load.

When, however, the armature is carrying current it exerts a magnetomotive force, called armature reaction, which combines with the magnetomotive force of the field winding and changes both the distribution and the total value of the flux entering the armature.

Fig. 137 (*a*) shows the distribution of flux produced by the armature m.m.f. acting alone, and the values of the m.m.f. of the

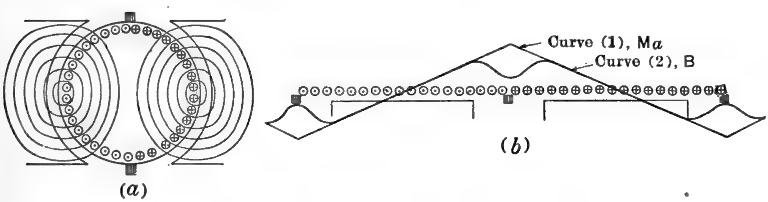


FIG. 137. Armature m.m.f. and flux.

armature at all points around the circumference are plotted in curve 1, Fig. 137 (*b*). The brushes are placed on the no-load neutral points. The armature m.m.f. M_a is a maximum in line with the brushes and falls off as a linear function to zero under the centre of the poles. The distribution of the flux produced by the armature m.m.f. is shown in curve 2, Fig. 137 (*b*).

Fig. 138 (*a*) represents the conditions when the m.m.f.'s of field and armature are acting together and with the brushes still on the

no-load neutral points. Curve 1, Fig. 138 (*b*), shows the resultant m.m.f. acting at each point. Its ordinates are represented by M and they are the sum of the corresponding ordinates M_f and M_a.

The m.m.f. across the pole face is no longer constant but is decreased over one half and increased over the other half by the same amount. The flux density B (curve 2) at each point is still proportional to the m.m.f. and inversely proportional to the

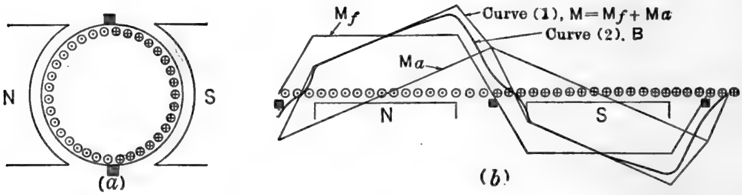


FIG. 138. Distribution of flux and m.m.f. under load, with the brushes on the no-load neutral points.

reluctance of the path, but, since part of the path is made up of a magnetic material, due to the effect of saturation, the increase of flux in one half of the pole is less than the decrease in the other half and consequently the total flux is decreased.

If it were not for the effect of saturation the ordinates of curve 2, Fig. 138 (*b*), could be obtained by adding the corresponding ordinates of Fig. 136 (*b*) and Fig. 137 (*b*).

The neutral points are no longer midway between the poles but have been shifted in the counter-clockwise direction in Fig. 138 (*a*) and to the right in Fig. 138 (*b*). To prevent sparking the brushes must be moved up to or a little beyond the load neutral points. In a generator the brushes must be moved forward in the direction of rotation and in a motor must be moved backward against the direction of rotation as indicated in Fig. 138.

With the brushes midway between the poles the direction of the armature m.m.f. is at right angles to the field m.m.f. and it therefore does not weaken it but only causes a distortion of the flux and a slight decrease due to saturation. In this case the m.m.f. of the armature is cross magnetizing.

When, however, the brushes are moved into the fringe of lines at the pole tips, as in Fig. 139 (*a*), the two m.m.f.'s are no longer at right angles and as seen in Fig. 139 (*b*) the part of the armature m.m.f. which is subtracted from the field m.m.f. is much greater

than the part which is added to it and therefore the resultant m.m.f. is reduced and the flux is both distorted and decreased.

Referring to Fig. 139 (*a*) the armature conductors may be separated into two groups; namely those between *a* and *d* included in the double angle of advance α with their return conductors from *c* to *b* and those under the pole between *b* and *a* with their return conductors between *d* and *c*. The first group acts in direct opposition

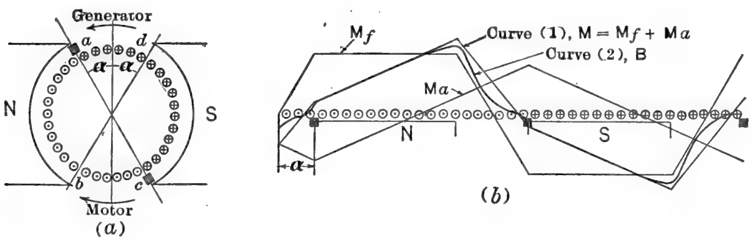


FIG. 139. Distribution of flux and m.m.f. under load, with the brushes under the pole tips.

to the field m.m.f. and decreases the flux crossing the air gap. They are therefore called the demagnetizing ampere turns of the armature. This demagnetizing m.m.f. increases as the shift of the brushes is increased and it also increases directly with the armature current.

The second group exerts a m.m.f. at right angles to the field m.m.f. and distorts the flux as in Fig. 137 (*a*) but only causes a slight decrease due to saturation. They are called the cross-magnetizing ampere turns of the armature.

For sparkless commutation without the use of interpoles the brushes must be moved ahead of the neutral points in order that the coils short circuited by them may be cutting the fringe of flux at the pole tips. E.m.f.'s are thus generated in the coils opposing the back e.m.f.'s due to inductance, and they aid in reversing the current. As the armature current is increased a point is finally reached where the armature m.m.f. is so strong that it overbalances the field m.m.f. at the pole tips and therefore no reversing field is left and commutation is not possible without interpoles. It is of no use to move the brushes further ahead because that only increases the demagnetizing component of armature m.m.f. and decreases the flux more. In direct-current machines without interpoles the armature ampere turns per pair of poles at full load

should not exceed 80 per cent of the field ampere turns per pair of poles.

Fig. 140 shows the effect of moving the brushes to the centre of the poles. The whole armature m.m.f. is demagnetizing and

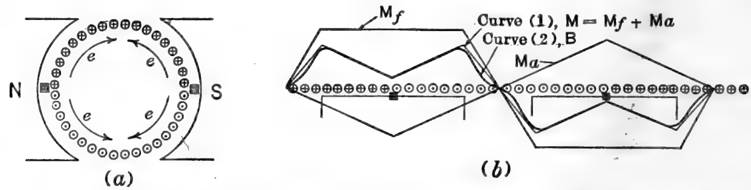


FIG. 140. Distribution of flux and m.m.f. with the brushes under the centres of the poles.

the flux is reduced to a small value. There is no difference of potential between the brushes since the sum of the e.m.f.'s generated in one half of the conductors in series between the brushes is exactly equal and opposite to that generated in the other half.

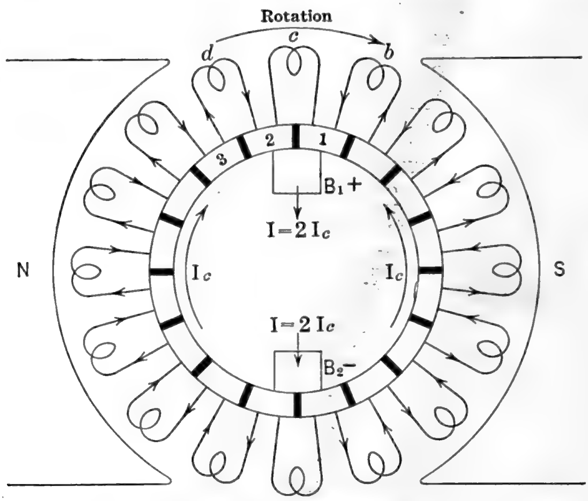


FIG. 141. Commutation.

110. Commutation. Commutation is the most important problem in direct-current machinery.

Fig. 141 represents the armature winding of a bipolar generator. The current entering by the brush B_2 divides into two

equal parts I_c which follow the two paths through the winding and unite again at the brush B_1. Any coil c while moving from B_2 to B_1 carries a current I_c. After passing B_1 it carries an equal current I_c but in the opposite direction and, therefore, while passing under the brush B_1 the current changes from I_c to $-I_c$, that is, it is commutated or reversed.

The factors which affect commutation are, I_c the intensity of the current to be commutated, T the time of commutation, r_c the resistance of the contact of the brush with the commutator, r the resistance of the armature coil short circuited, L the self-inductance of the coil, and finally the direction and intensity of flux cut by the coil during commutation.

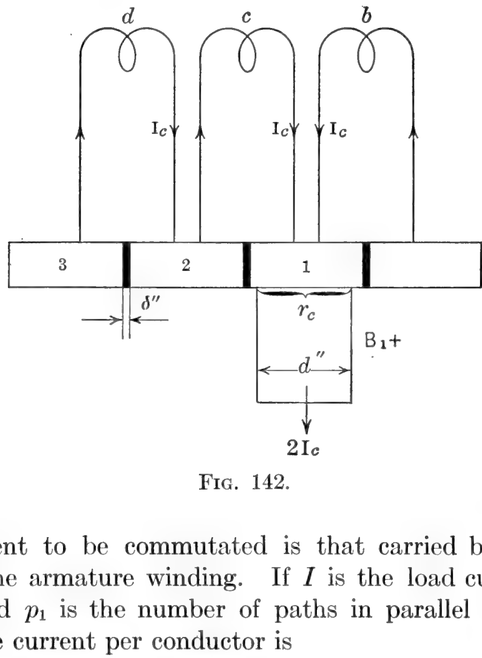


FIG. 142.

The current to be commutated is that carried by each conductor of the armature winding. If I is the load current of the machine and p_1 is the number of paths in parallel through the winding, the current per conductor is

$$I_c = \frac{I}{p_1},$$

and increases directly as the load current.

The time of commutation is the time during which two adjacent commutator bars are short circuited by the brush. In Fig. 142 commutation of the current in coil c begins as soon as the brush

touches bar 2 and must be completed when the brush breaks contact with bar 1. If the width of the brush is d ins., the thickness of insulation between bars is δ ins., and the peripheral speed of the commutator is V ins. per sec., the time of commutation is

$$T = \frac{d - \delta}{V} \text{ sec.}$$

Since δ is very small, the time of commutation varies directly as the width of the brush and inversely as the speed of the machine.

The resistance of the brush contact plays a very important part in commutation; it tends to reduce the current in the short-circuited coil to zero and then to build it up in the opposite direction. It would produce complete commutation if it were not opposed by the effects of the resistance and self-induction of the coil. Its function is illustrated in Fig. 143. If the resistance of the total

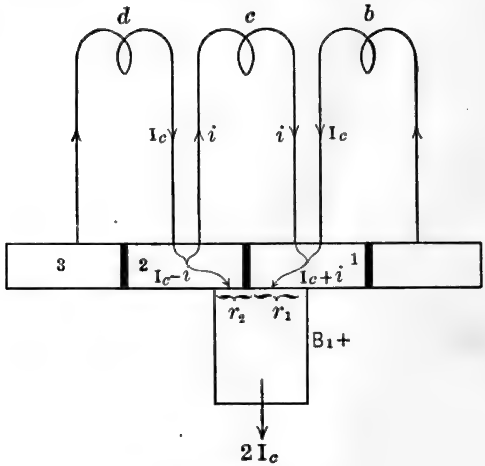


Fig. 143.

brush contact is r_c, then in Fig. 142 the drop of potential between the brush B_1 and bar 1 is $2I_cr_c$. As soon as the brush touches bar 2 commutation begins and the brush-contact resistance must be separated into two parts, r_1 the resistance from the brush to bar 1 and r_2 the resistance from the brush to bar 2. If at the instant represented in Fig. 143 the current in the coil is i, then the current

flowing from bar 1 to the brush is $I_c + i$ and the drop of potential is $(I_c + i)\, r_1$; the current from bar 2 to the brush is $I_c - i$ and the drop of potential is $(I_c - i) r_2$. Since the resistance r_1 is increasing while r_2 is decreasing, the current from bar 2 will increase while that from bar 1 will decrease and the current in the coil will decrease. Neglecting the resistance and self-induction of the coil the current flowing in the coil will be zero when r_1 and r_2 are equal and when therefore half of the time of commutation has passed. Any further increase in the resistance r_1 will cause part of the current from coil b to flow through coil c in order to reach the brush B_1 by the path of least resistance. As the resistance r_1 still increases, more and more current flows through c until r_1 becomes infinite as the brush breaks contact with bar 1 and the total current I_c from b flows through c. Commutation is then complete.

In Fig. 145, curve (1), the current in coil c is plotted on a time base for half of one revolution; it is reversed in the time T, represented by OT, during which the coil moves across the brush B_1, and it must vary according to a straight line law. This can be proved as follows:

If Fig. 143 represents the condition t seconds after the beginning of commutation, neglecting the resistance and self-inductance of the coil, the drop of potential from the commutator to the brush must be the same at both sides, or

$$(I_c + i)\, r_1 = (I_c - i)\, r_2;$$

but

$$r_1 = r_c \frac{T}{T - t} \quad \text{and} \quad r_2 = r_c \frac{T}{t},$$

therefore,

$$(I_c + i)\, r_c \frac{T}{T - t} = (I_c - i)\, r_c \frac{T}{t}. \qquad . \quad . \quad . \quad (203)$$

Solving for i this gives

$$i = I_c \frac{T - 2\,t}{T}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204)$$

which is the equation of a straight line.

When $t = \frac{T}{2}$, $i = 0$, and when $t = T$, $i = -I_c$.

If the resistance of the coil is taken into account, the drop of potential across r_2 (Fig. 144) must be greater than the drop

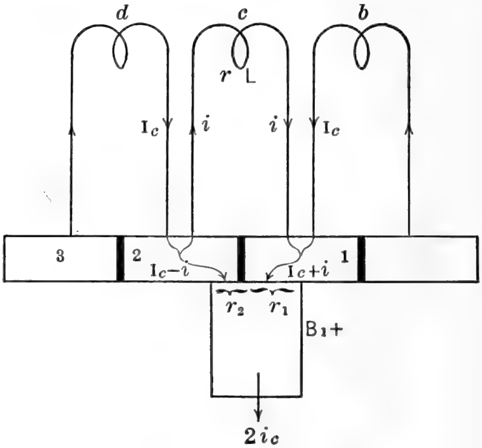


Fig. 144.

across r_1 by the amount required to maintain the current i through the resistance r; therefore,

$$(I_c + i)\, r_c \frac{T}{T - t} + ir = (I_c - i)\, r_c \frac{T}{t},$$

and

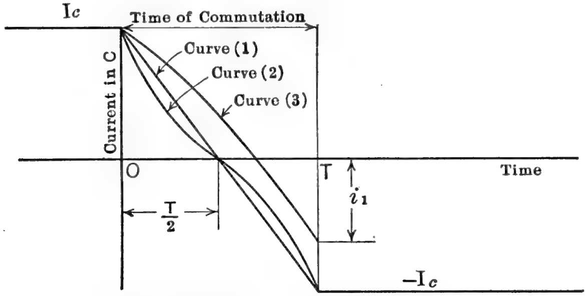
$$i = I_c \frac{r_c\,(T^2 - 2Tt)}{r\,(Tt - t^2) + r_c T^2}. \quad . \quad . \quad . \quad . \qquad (205)$$

When $t = \frac{T}{2}$, $i = 0$, and when $t = T$, $i = -I_c$.

The current therefore passes through zero at the same instant as before and is completely reversed in the same time, but the variation does not follow a straight line law but a curve as shown in curve (2), Fig. 145. The effect of the coil resistance is very small and may be neglected.

The effect of the self-inductance of the armature coil must next be considered. Armature coils are partially surrounded by iron and therefore have a large self-inductance, which is proportional to the square of the number of turns in the coil. With full-pitch drum windings both the coils in one slot will be short circuited at one time and the inductive flux linking with each of them will be

almost twice as large as in the case of fractional pitch windings. This is partly a mutual inductance effect but may all be included under self-inductance. (See Fig. 146.)



(1) Commutation by contact resistance only
(2) Effect of the resistance of the coil
(3) Effect of the inductance of the coil

FIG. 145.

When the current in a coil of self-inductance L henrys is changing at the rate $\frac{di}{dt}$ amperes per second, an e.m.f. $L\frac{di}{dt}$ volts is generated in a direction opposing the change of current.

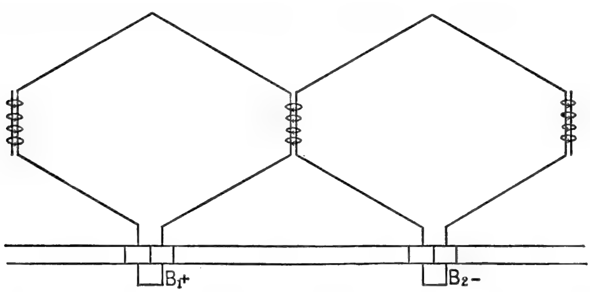


FIG. 146. Inductive flux in a full-pitch drum winding.

In Fig. 144 the drop of potential from bar 2 to the brush is the same by the two paths, one through the resistance r_2 and the other through the coil in series with the resistance r_1; and thus

$$(I_c + i)\, r_1 + ri + L\frac{di}{dt} = (I_c - i)\, r_2,$$

or, substituting the values of r_1 and r_2 found above,

$$(I_c + i)r_c \frac{T}{T - t} + ri + L\frac{di}{dt} = (I_c - i)\frac{T}{t},$$

and

$$L\frac{di}{dt} + i\left\{r + \frac{r_c T^2}{t(T - t)}\right\} + \frac{r_c I_c T(2t - T)}{t(T - t)} = 0. \quad . \quad (206)$$

This equation cannot be solved easily. Due to the effect of self-inductance the current i does not decrease so quickly as in curve (1) but follows curve (3), shown in Fig. 145. It will not have fallen to zero at time $t = \frac{T}{2}$ and will have grown to a value i_1 less than I_c at time $t = T$. Commutation will therefore not be complete and when the brush breaks contact with bar 1 the current $I_c - i_1$ from coil b will try to jump across to the brush and will cause a spark.

In order to reverse the current completely in the time T it is necessary to have an e.m.f. generated in the coil to assist commutation. The brushes of a generator are therefore moved ahead in the direction of rotation so that the coil when short circuited is cutting the fringe of lines from the pole tip.

The intensity of this field is not constant but increases as the conductor approaches the pole tip; therefore, the e.m.f. produced by it also varies.

If at the time represented in Fig. 144 there is an e.m.f. e generated in the coil assisting commutation, equation 206 can be written

$$e + L\frac{di}{dt} + i\left(r + \frac{r_c T^2}{t(T - t)}\right) + \frac{r_c I_c T(2t - T)}{t(T - t)} = 0. \quad . \quad (207)$$

This equation cannot be solved in general but it is possible to determine the value of e required at any instant to cause the current to vary as a linear function of time from I_c to $-I_c$ in the time T.

On this assumption

$$i = I_c\frac{T - 2t}{T}$$

and

$$\frac{di}{dt} = -\frac{2I_c}{T}.$$

Substituting these values in equation gives

$$e - I_c\left\{\frac{2L}{T} - r\left(1 - 2\frac{t}{T}\right)\right\} = 0,$$

or

$$e = I_c\left\{\frac{2L}{T} - r\left(1 - 2\frac{t}{T}\right)\right\}, \quad \ldots \ldots \quad (208)$$

which gives at the beginning of commutation $t = 0$,

$$e_0 = I_c\left(\frac{2L}{T} - r\right), \quad \ldots \ldots \quad (209)$$

and at the end of commutation $t = T$,

$$e_T = I_c\left(\frac{2L}{T} + r\right). \quad \ldots \ldots \quad (210)$$

This e.m.f. is proportional to the current I_c but is independent of the brush resistance r_c.

If the e.m.f. generated in the coil is less than that required to reverse the current completely in time T, commutation is imperfect and there is a tendency to spark, and if the e.m.f. is so large that the current is more than reversed there is a tendency to spark due to overcommutation.

The contact resistance helps to prevent sparking when the e.m.f. generated in the coil by rotation is either too great or too small to produce perfect commutation.

When commutation is produced by the high-resistance brush contact without the aid of any e.m.f. generated in the coil, it is called "natural" or "resistance" commutation; when it is assisted by an e.m.f. generated in the coil, it is called "forced" or "voltage" commutation.

Resistance commutation can never be perfect unless the self-inductance of the coil is negligible, but at light loads it will reverse the current without injurious sparking. Assume that the brushes of a generator delivering half load are set on the corresponding neutral line and that commutation is satisfactory. If the load is increased the increased m.m.f. of the armature causes the neutral line to move ahead so that the coil short circuited is cutting a field of such a direction as to tend to maintain the current or even to increase it. The reversal of the current is therefore retarded

and there is a greater tendency to spark than before. If the load is reduced the neutral line falls behind the brushes and a voltage assisting commutation is generated in the coil.

If the load is all removed from a generator when the brushes are not on the no-load neutral there will be sparking due to the fact that the short-circuited coil has an e.m.f. generated in it which builds up a current i and stores energy $L\frac{i^2}{2}$ in the field. This energy appears as a spark.

Voltage commutation is also limited in its application and as the current in the armature is increased a point is reached (usually about twenty-five per cent overload) beyond which sparkless commutation is impossible, since when the current is increased a stronger field is required to reverse it, but the stronger current in the armature increases the m.m.f. of the armature and moves the neutral line ahead of the brushes and at the same time decreases the flux. The brushes have to be advanced further and the demagnetizing effect is increased. When the armature m.m.f. is large enough to overbalance the field m.m.f. the flux at the pole tip is wiped out and voltage commutation is impossible. Moving the brushes further ahead only decreases the flux.

To take full advantage of voltage commutation it would be necessary to vary the position of the brushes with varying load but this is not practicable, and therefore the brushes must be set to give good commutation at some intermediate load and the resistance of the brush contact must be relied on to prevent sparking above and below this point. Modern machines are designed to give good commutation at all loads from no load to twenty-five per cent overload with fixed brush position.

111. Interpoles. Interpoles are small poles placed midway between the main poles of either motors or generators. They are magnetized by a winding connected in series with the armature and carrying the load current. It has been seen that the brushes of motors must be moved back under load to obtain satisfactory commutation. This is possible with motors which run only in one direction, but with all reversible motors the brushes must be fixed on the no-load neutral points. Perfect commutation under these conditions was not possible until the introduction of the interpole or commutating pole. Fig. 147 shows an interpole motor or generator. The brushes are fixed on the no-load neutral points.

The interpoles have the same effect as moving the brushes since they move the poles magnetically.

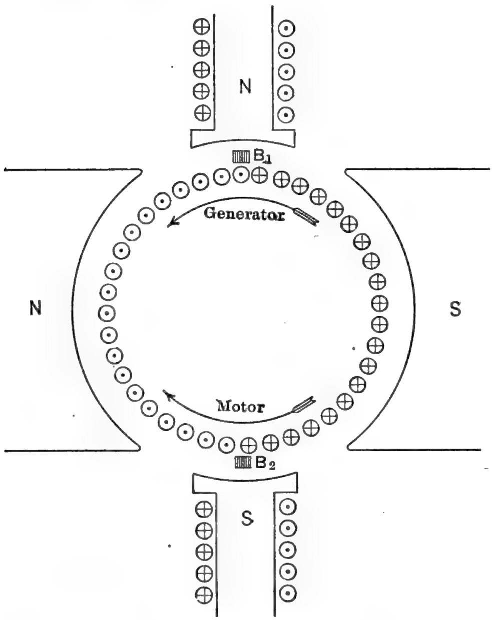


FIG. 147. Interpole generator or motor.

The m.m.f. of the interpole winding must oppose the m.m.f. of the armature and must be strong enough to overbalance it and produce a field under the interpole of the proper intensity to reverse

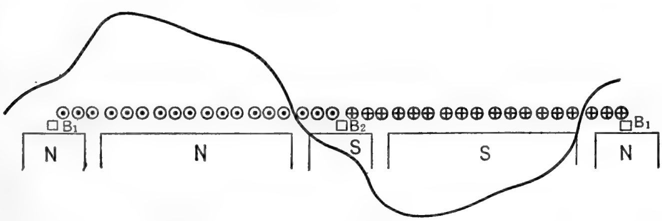


FIG. 148. Flux distribution in interpole generator or motor.

the current in the short-circuited coil. Since the interpole winding is in series with the armature the commutating field increases with load and satisfactory commutation up to and beyond the

overload limits of output set by armature heating can be obtained.

Interpoles are not so necessary on generators as on motors but they are very useful in the case of a generator which supplies a rapidly fluctuating load since the interpole m.m.f. follows exactly the fluctuations of the armature m.m.f. and so prevents sparking. Fig. 148 shows approximately the flux distribution in an interpole generator or motor.

112. Sparking. When a current i amperes flows in a coil of inductance L henrys, energy is stored in the magnetic field surrounding the coil of value

$$\omega = L\frac{i^2}{2} \text{ watt-seconds.}$$

At the beginning of commutation the current in coil c is I_c and the energy stored in its field is $L\frac{I_c^2}{2}$ watt-seconds. While the current is falling to zero this amount of energy must either be given back to the electric circuit as useful work or wasted as heat in the resistance of the short-circuited coil. At the end of commutation the same amount of energy $L\frac{I_c^2}{2}$ must be stored in the field by a current I_c flowing in the opposite direction in order that the current from coil b may flow freely through coil c, and that there may be no tendency to spark.

If when the brush B_1 (Fig. 144) breaks contact with bar 1 the current has only decreased to a value i, the energy $L\frac{i^2}{2}$ stored in the field of the coil will appear as a spark and the current I_c from the coil b, since it cannot immediately flow through the coil c against the inertia of its magnetic field, will try to follow the brush and will produce an arc which will increase the sparking. Behind this arc is the energy stored in the magnetic field of all the coils on one side of the armature, since any decrease in the current in coil b is accompanied by a decrease in all the coils in series with it. This energy is not, however, available instantaneously due to the inertia of the magnetic field, and commutation is complete before the current has time to decrease appreciably. At the instant of breaking contact with bar 1 the current density in the brush tip and in the edge of the bar is very high and a very high temperature will be produced locally which may volatilize a small amount

of copper. If this condition is allowed to continue the commutator will become roughened and the brushes will be gradually destroyed.

If the current in coil c has just reached zero when the brush breaks contact, there will be no spark due to energy stored in the field of c but that due to the current from b will remain.

If an e.m.f. had been present in the coil in the direction necessary to reverse the current and of such strength that the current had reached a value i greater than I_c, then at the instant of breaking contact the excess energy $L\frac{i^2}{2} - \frac{I_c^2}{2}$ would tend to produce a spark.

113. Voltage Characteristic or Regulation Curve. The voltage characteristic of a direct-current generator is the relation between the terminal e.m.f. and the current output.

The e.m.f. generated in the armature is

$$\mathcal{E} = Zn\Phi \frac{p}{p_1} 10^{-8} = Kn\Phi \text{ volts.} \quad \text{(Art. 103.)}$$

Take the case of a separately excited generator where the speed n and the field current I_f are both kept constant.

At no load the flux crossing the air gap under each pole is Φ_0 and the e.m.f. generated is

$$\mathcal{E}_0 = Kn\Phi_0;$$

this is also the terminal e.m.f. at no load.

As the generator is loaded the terminal e.m.f. decreases due to two causes, (a) armature reaction and (b) armature resistance.

(a) When current flows in the armature, the armature m.m.f. decreases the flux crossing the air gap and therefore decreases the generated e.m.f.

This has been shown in Art. 109. If the brushes are moved ahead in a generator or back in a motor in order to obtain satisfactory commutation, the m.m.f. of the armature turns between the poles opposes the field m.m.f. and therefore decreases the flux. These are the demagnetizing ampere turns. The m.m.f. of the turns under the poles distorts the flux and causes a slight decrease due to the high saturation of one half of the pole tips. They are called the cross-magnetizing ampere turns. These two effects are combined under the term armature reaction.

Thus armature reaction is due to the m.m.f. of the armature currents and causes a decrease of the flux crossing the air gap and therefore a decrease in the generated e.m.f. The armature m.m.f. and the drop in e.m.f. caused by it increase as the load current increases.

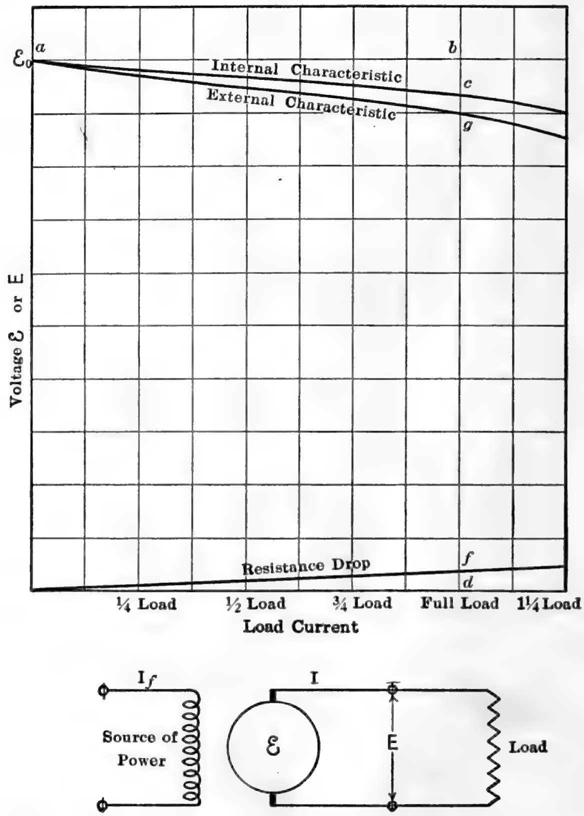


FIG. 149. Voltage characteristics of a separately excited generator.

(b) It is necessary to distinguish between the e.m.f. $\mathcal{E}$ generated in the armature and the terminal e.m.f. E. At no load they are the same but when current flows in the armature part of the generated e.m.f. is consumed in driving the armature current I through the resistance of the armature winding and brushes r. This resistance drop is Ir and increases directly with the current.

The terminal e.m.f. is $E = \mathcal{E} - Ir$. . . (211)

In Fig. 149 *ac* shows the relation between the generated e m.f. $\mathcal{E}$ and the armature current I. It is called the internal characteristic of the generator. *bc* is the drop in generated e.m.f. at full load due to armature reaction. The ordinates of *of* represent the e.m.f. consumed by the armature resistance. The ordinates of *ag* are the differences between the corresponding ordinates of *ac* and *of*.

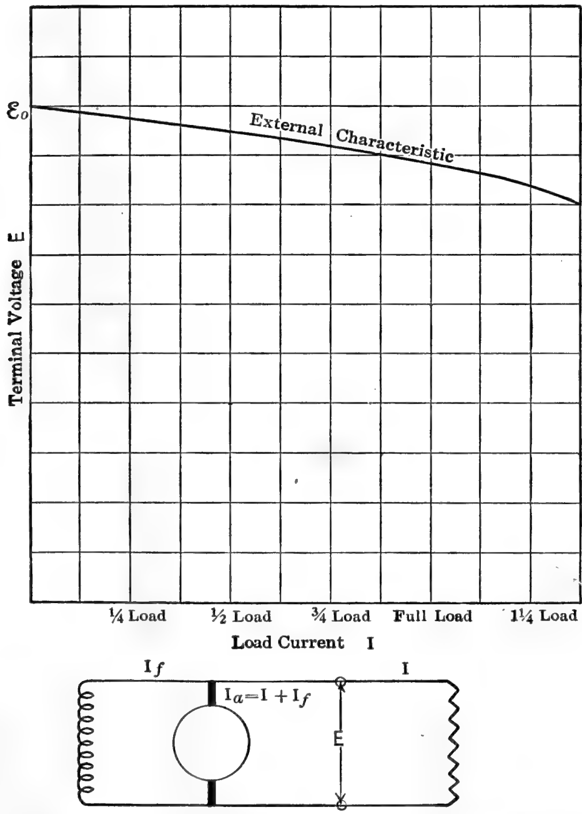


FIG. 150. Voltage characteristic of a shunt generator.

ag therefore shows the relation between the terminal e.m.f. E and the armature current and is called the external voltage characteristic or regulation curve of the generator.

114. Regulation. The regulation of a generator is defined as the rise in voltage when full load is thrown off expressed as a per cent of full-load voltage.

If od represents full-load current, gd is the terminal voltage at full load and bg is the rise in voltage that would occur if the load were removed; therefore the regulation is $\frac{bg}{gd}100$ per cent.

So far only the separately excited generator has been considered. In the shunt generator, armature reaction and armature resistance cause a decrease in terminal e.m.f. under load, but a third condition must also be taken into account. The field circuit is connected across the armature terminals and the current in it is proportional to the terminal e.m.f. Thus, when the terminal e.m.f. decreases due to armature reaction and armature resistance, the field current also decreases and causes a further decrease in the flux and therefore the terminal e.m.f. of a shunt-excited generator is less than it would be if the machine were separately excited. (Fig. 150.)

The load current can be increased by decreasing the resistance in the load circuit up to the point M in Fig. 151 which is the com-

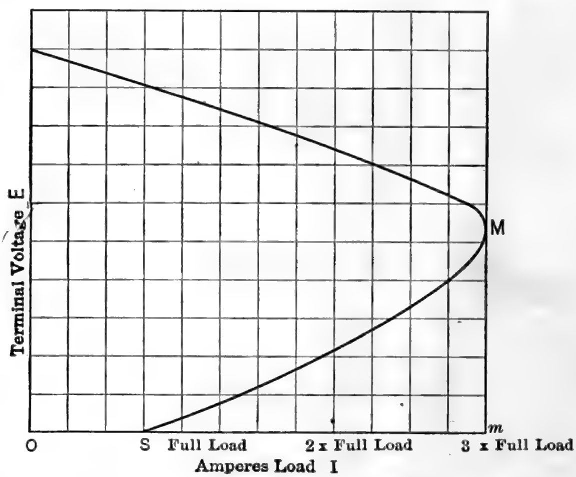


FIG. 151. Voltage characteristic of a shunt generator.

plete voltage characteristic of a shunt generator. om is the maximum current output. If the load resistance is decreased further, the armature current increases for an instant and then decreases as its m.m.f. wipes out part of the flux and causes the generated e.m.f. to decrease. Finally when the load resistance is zero and

the generator is short circuited, the flux is reduced to such a value that the generated e.m.f. is only large enough to supply the resistance drop in the armature. The armature current is then *os* and the terminal e.m.f. is zero. Rated

The maximum current *om* is many times full-load current and can only be reached with small machines of bad regulation.

115. Field Characteristic. The field characteristic of a generator is the curve showing the variation of field current with load current to maintain a constant terminal voltage at constant speed.

In a self-excited generator, Fig. 152, the terminal voltage can be maintained constant as the load current increases if the field

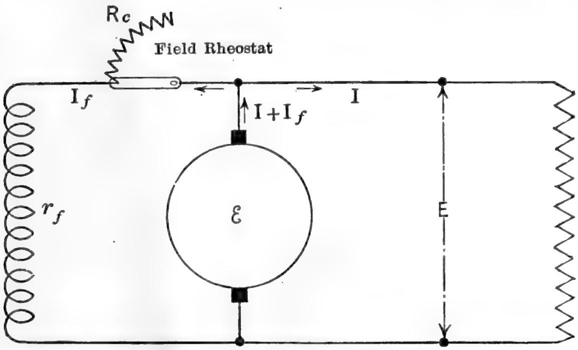


FIG. 152. Shunt generator.

current is increased to such a value that the increase in field m.m.f. will not only overcome the effect of armature m.m.f. but will produce an increase in flux to provide the extra e.m.f. to supply the armature resistance drop.

In Fig. 153 *ad* is the field characteristic of the generator, Fig. 152, *oa* is the field current required to produce the rated terminal voltage E at no load. When a load current *og* is supplied by the generator a larger field current *oc* is required. The increase in field current $i = ac$ exerts a m.m.f. which overcomes the effect of the armature m.m.f. and produces an increase in the flux to provide for the armature resistance drop Ir. The generated voltage is

$$\mathcal{E} = E + Ir. \qquad (212)$$

Fig. 154 shows the saturation curve of the generator. *od* is the e.m.f. generated at no load by the field current $I_f = oa$ and it is

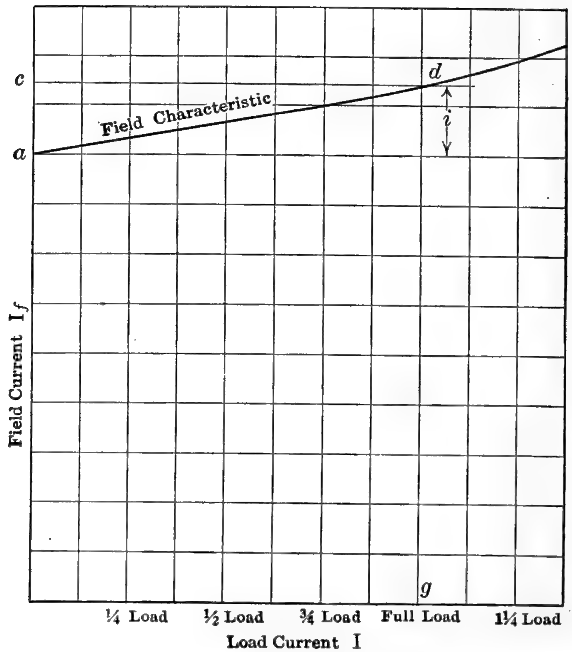


FIG. 153. Field characteristic.

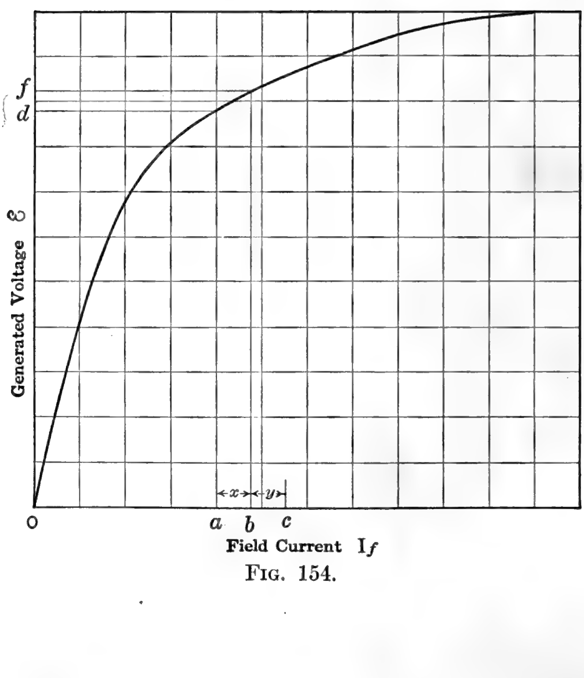


FIG. 154.

$= E$. of is the voltage which must be generated in the armature when supplying full-load current $I = og$. df is the increase in generated e.m.f. required to overcome the armature resistance drop Ir; this requires an increase in field current $ab = x$. The other component of field current $bc = y$ is required to overcome the effect of the armature m.m.f.

The total increase in field current required to maintain constant terminal voltage is

$$ac = x + y = i. \quad \text{(Figs. 153 and 154.)}$$

If N is the number of turns on the field winding the increase in field m.m.f. is Ni ampere turns.

The increase of field current under load is obtained by gradually cutting out resistance from the field rheostat R_c in series with the field winding.

This regulation must be done by hand and cannot take care of sudden changes in load.

116. Compound Generator. Increase of field m. m. f. under load can be obtained automatically by placing a series winding on the field poles in addition to the shunt winding. The series winding carries the load current and so the field m.m.f. increases under load.

Such a generator is called a compound-wound generator. If the series winding is designed so that the terminal voltage is the same at full load as at no load the generator is flat-compounded; if the terminal voltage at full load is higher than at no load the generator is over-compounded.

In Fig. 155 are shown the regulation curves or voltage characteristics of a machine (1) self-excited, (2) separately excited, (3) flat-compounded and (4) over-compounded. If a generator is flat-compounded so that it gives the same terminal voltage at full load as at no load it will be slightly over-compounded below full load.

If the field characteristic of a self-excited generator is known (Fig. 153) and the number of turns on the field winding is N, the number of turns required on the series field to produce flat-compounding can easily be calculated. The increase in field m.m.f. at full load is Ni ampere turns where i is the increase in field current from no load to full load. Since this m.m.f. is to be produced by the load current I, the number of turns required is $N_s = \frac{Ni}{I}$.

The compounding of a machine can be varied by connecting a resistance in shunt to the series winding as shown at S in Figs. 156 and 157.

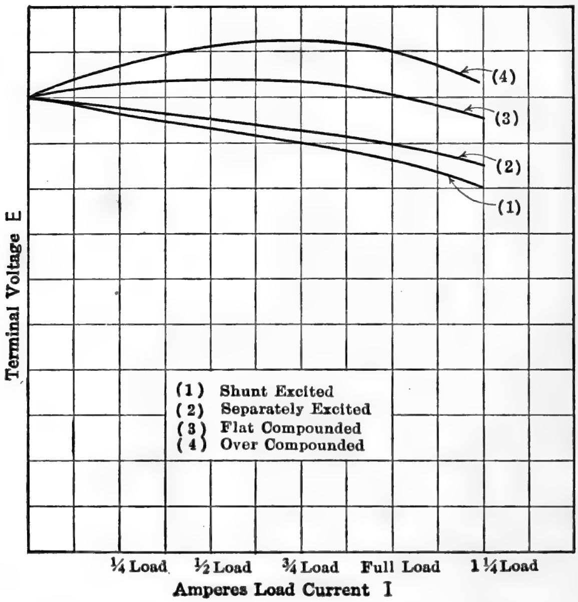


FIG. 155. Voltage characteristics.

In a generator supplying a rapidly fluctuating load the shunt to the series winding must be designed with its inductance in the same ratio to the inductance of the series winding as its resistance

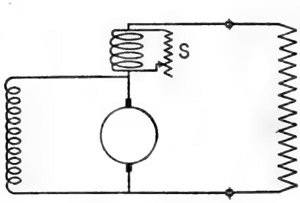


FIG. 156. Compound generator (short shunt).

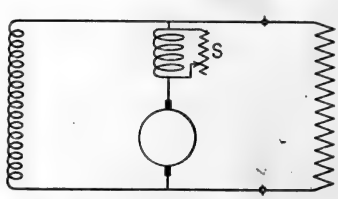


FIG. 157. Compound generator (long shunt).

is to the resistance of the series winding in order that the variable current may divide up in the correct proportions to give the required compounding.

In compound-wound machines the shunt winding may be connected across the armature terminals, Fig. 156, called short shunt, or outside of the series winding, Fig. 157, called long shunt. The characteristic curves are not affected to a great extent by the dif-

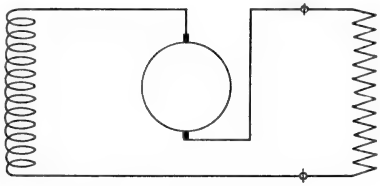


FIG. 158. Series generator.

ference in connection since with the short shunt the voltage across the shunt winding is higher than with the long shunt by the resistance drop in the series winding and the current in the series winding is less by the amount supplied to the shunt winding.

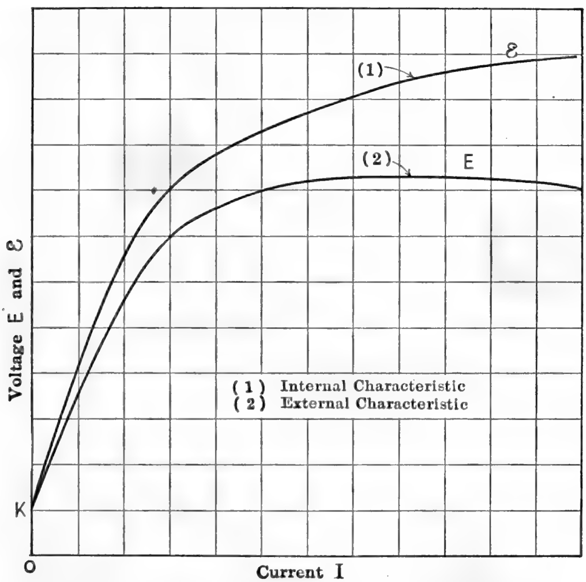


FIG. 159. Voltage characteristics of a series generator.

117. Series Generator. The series generator, Fig. 158, is excited by a series winding carrying the load current and has no shunt winding.

Its voltage characteristics are shown in Fig. 159. At no load

there is no current in the field winding and the only voltage generated is that due to the residual magnetism. It is shown as *ok*. As the load current increases the flux increases and the generated e.m.f. increases until the magnetic circuit becomes saturated and the decreasing permeability and increasing leakage flux cause the generated e.m.f. to fall off. This is shown in curve (1) which is the internal characteristic. The ordinates of the external characteristic or regulation curve (2) are less than those of (1) by the resistance drop in the armature and series field.

The terminal voltage of a series generator can be varied by connecting a resistance in shunt to the field winding.

118. Parallel Operation. In power houses in which the load varies at different hours of the day a number of generators are usually installed. When the load is light one generator is operated and supplies the demand and when the load increases a second machine is started up and connected in parallel with the first and its excitation is adjusted until it takes its proper share of the load.

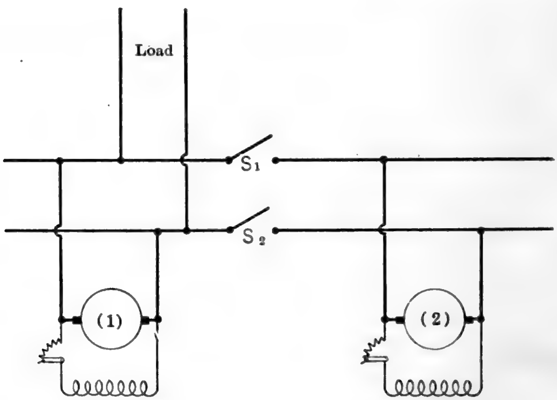


FIG. 160. Parallel operation of shunt generators.

Fig. 160 shows two shunt generators (1) supplying power and (2) ready to be connected in parallel with it. Before closing the switches S_1 and S_2 which connect the second machine to the load it is necessary that its polarity be correct and that its terminal voltage be the same or a little higher than that of (1).

If the field rheostat of (2) is so adjusted that the voltage of (2) is the same as the voltage of (1) and switch S_1 is closed, then, if there is no voltage across S_2, it may be closed. But if the voltage

across S_2 is found to be about double the terminal voltage, the field of (2) must be reversed before closing switch S_2. After closing S_2 the field rheostat of (2) must be adjusted until (2) takes its proper share of the load. If the voltage of (2) is the same as the voltage of (1) when the switch is closed, (2) will not take any load but will run idle. If the voltage of (2) is less than the voltage of (1), machine (2) will run as a motor driving its prime mover and will draw power from (1). If, however, the terminal voltage of (2) is higher than that of (1), machine (2) will supply part of the load and will relieve (1) until the voltages of the two

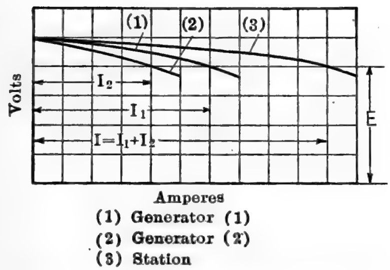


FIG. 161. Voltage characteristics.

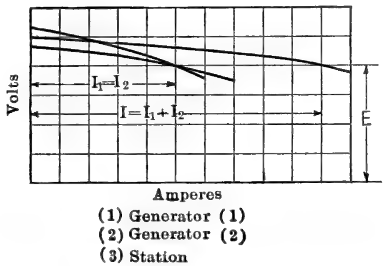


FIG. 162. Voltage characteristics.

are the same. Fig. 161 represents the voltage characteristics of the two machines plotted on the same base. If the terminal voltage is E (1) supplies a current I_1 and (2) a current I_2 and the total current supplied by the station is $I = I_1 + I_2$. The machine with the flatter characteristic will supply the greater amount of power. If the two machines are rated at the same current output, (2) can be made to take its share of the load by cutting out resistance from its field rheostat and so raising its voltage characteristic and inserting resistance in the field circuit of (1) and lowering its characteristic as shown in Fig. 162.

Shunt generators will operate in parallel and divide up the load in proportion to their capacities if their voltage characteristics are similar, that is, if their terminal voltage falls from no load to full load by the same amount and in the same manner. If the characteristics are different a proper division of load can be obtained by regulating the field rheostats.

119. Parallel Operation of Compound Generators. Fig. 163 shows two compound-wound generators connected in parallel and supplying power to a load circuit. Their voltage characteristics

are shown in Fig. 164. Assume that the prime mover of (1) runs for an instant at a slightly increased speed; the voltage of (1) rises and it takes more than its share of the load; the voltage of (2) falls because its load is decreased and its series excitation is decreased. Machine (1) therefore takes more of the total load and its voltage rises higher until it supplies all the load and in addition drives (2) as a motor. Since the current in (2) is reversed the m.m.f. of its series winding is also reversed and it runs

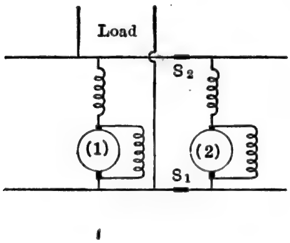


FIG. 163.

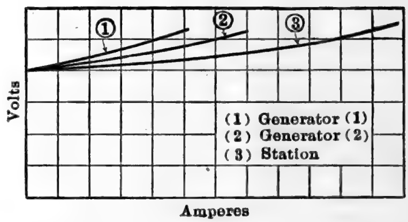


FIG. 164. Voltage characteristics.

as a differential motor driving its prime mover at a high speed until the load on (1) becomes so great that the protective apparatus opens the circuit and shuts down the system.

To get over this difficulty the equalizer connection *ee*, Fig. 165, is used. It is a conductor of low resistance connecting in multiple the series windings of the two machines. Now if the prime mover of (1) runs above normal speed the voltage of (1) rises and it takes an increased load. The increase of current does not all go through the series winding of (1) but divides between the windings of (1) and (2) in inverse proportion to their resistances and so prevents any decrease of the voltage of (2). Thus with an equalizer connection (2) will still hold its load. The resistances of the series windings must be adjusted so that the load current will divide between them in such proportion that each machine will supply its proper share of the load.

Before connecting machine (2) in parallel with (1) which is delivering power, first close switches S_2 and S_3, Fig. 165, and adjust the shunt field of (2) until its terminal voltage is the same as that of (1). The excitation of (2) is now provided partly by its shunt field and partly by its series field carrying part of the load current. After checking the polarity to see that it is correct close

switch S_1 and adjust the shunt field of (2) until the machines divide the load in proportion to their capacities.

From the above discussion it is seen that two compound-wound generators connected in parallel form an unstable system unless an equalizer connection is placed between their series windings.

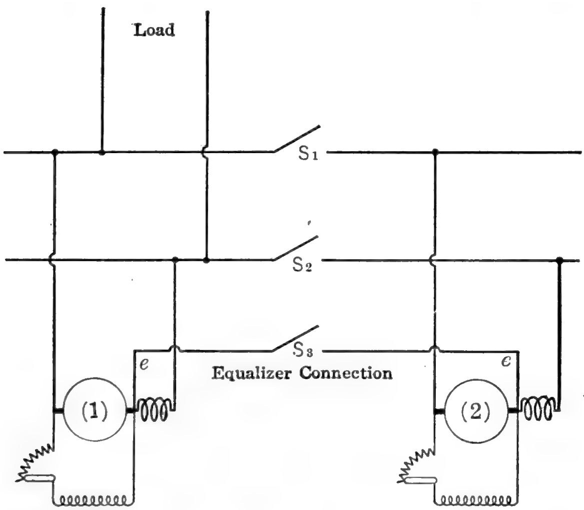


FIG. 165. Parallel operation of compound generators.

120. Tirrill Regulator. The Tirrill regulator is an automatic voltage regulator designed to maintain a steady voltage at the terminals of a direct-current generator irrespective of ordinary load fluctuations or changes in generator speed. It can also be made to compensate for line drop by increasing the generator voltage as the load increases.

The regulator controls the voltage by rapidly opening and closing a shunt circuit across the field rheostat of the generator. The rheostat is so adjusted that when in circuit it tends to reduce the voltage considerably below normal and when short circuited the voltage tends to rise above normal. The relative lengths of time during which the short circuit is closed or opened determines the average value of the field current and therefore the value of the terminal voltage.

The method of operation of the regulator is illustrated in Fig. 166. The regulator consists essentially of two magnets con-

trolling two sets of contacts. The main control magnet has two independent windings, one, the potential winding, connected across the generator terminals and the other across a shunt in the load circuit. The latter is the compensating winding and is only used when a rise of voltage with load is required. The relay magnet

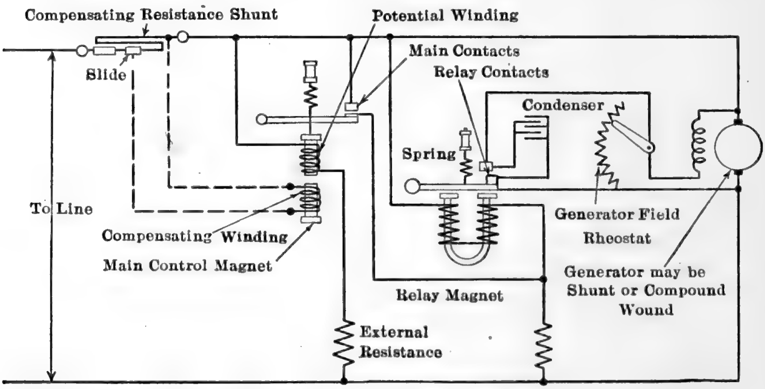


FIG. 166. Automatic voltage regulator.

is differentially wound and controls the circuit shunting the field rheostat. The operation is as follows: When the short circuit across the field rheostat is opened the voltage tends to fall below normal. The main control magnet is weakened and allows the spring to pull out the movable core until the main contacts are closed. This closes the second circuit of the differential relay and demagnetizes it. The relay spring then lifts the armature and closes the relay contacts. The field rheostat is short circuited and the field current and terminal voltage tend to rise. The main control magnet is strengthened and opens the main contacts allowing the differential relay to open the short circuit across the field rheostat. The terminal voltage falls again and this cycle of operations is repeated at a very rapid rate maintaining a steady voltage at the generator terminals. When the compensating winding is not used the terminal voltage is maintained constant.

When it is necessary to compensate for line drop and maintain a constant voltage at the receiver end of the line, the compensating winding is connected across a shunt in the load circuit. The resistance of the shunt is adjusted to give the required compound-

ing. The compensating winding opposes the action of the potential winding on the main control magnet so that as load increases a higher potential is necessary at the generator terminals in order to close the main contacts and open the shunt across the field rheostat. Thus the generator voltage rises with load. The condenser connected across the relay contacts serves to reduce the sparking when the circuit is opened.

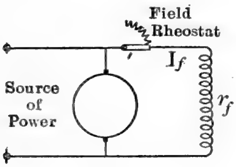


FIG. 167. Shunt motor.

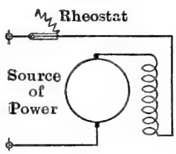


FIG. 168. Series motor.

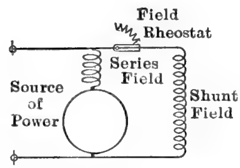


FIG. 169. Compound motor.

121. Electric Motors. In generators mechanical power is supplied and electrical power is generated. The speed is fixed by the prime mover and is constant. The terminal voltage is approximately constant in the shunt generator and flat-compound generator and increases with load in the over-compound generator and the series generator. The generated voltage is always greater than the terminal voltage by the drop in the armature resistance; it is

$$\mathcal{E} = E + Ir.$$

In motors electrical power is supplied and mechanical power is generated. The impressed e.m.f. is fixed by the supply circuit and is constant. The speed is either approximately constant as in the shunt motor or decreases with load as in the compound motor and series motor. The voltage generated in the armature has the same equation as the voltage in a generator, but it is a back voltage and opposes the current; the impressed voltage E is greater than the back-generated voltage by the armature resistance drop; thus,

$$E = \mathcal{E} + Ir, \qquad (213)$$

or

$$\mathcal{E} = E - Ir. \qquad (214)$$

122. Types of Motors. There are three types of direct-current motors corresponding to the three types of generators, shunt, compound and series. The shunt motor has its field circuit con-

nected across the line in shunt with the armature and therefore has a constant excitation. (Fig. 167.) The compound motor has a series winding carrying the load current in addition to its shunt winding. The excitation therefore increases with load. (Fig. 169.)

The series motor has its field circuit in series with the armature and has no shunt winding. Its excitation is zero at no load and increases directly with the load current. (Fig. 168.)

123. Speed Equation of a Motor. The impressed voltage is

$$E = \mathcal{E} + Ir,$$

and

$$\mathcal{E} = Zn\Phi \frac{p}{p_1} 10^{-8} = Kn\Phi \text{ volts;}$$

therefore

$$E = Kn\Phi + Ir$$

and

$$n = \frac{E - Ir}{K\Phi}; \qquad (215)$$

this is the speed equation.

The term Ir is small in comparison to E and can be neglected except at heavy loads.

The speed equation can therefore be written

$$n = \frac{E}{K\Phi}; \qquad (216)$$

thus, the speed of a motor is directly proportional to the impressed voltage and is inversely proportional to the flux crossing the air gap.

124. Methods of Varying Speed. The speed can be varied in three ways, (1) by varying the impressed voltage E, (2) by varying the flux Φ by reducing the field current, (3) by shifting the brushes.

(1) The voltage impressed on the armature can be varied by introducing a resistance R, Fig. 170, in series with the armature. If E_s is the supply voltage, the voltage impressed on the motor is

$$E = E_s - IR.$$

By increasing R, E can be reduced to any required value and any speed from standstill to normal speed can be attained but no increase of speed above normal.

Fig. 171 shows n as a function of E; the locus is a straight line passing through the origin.

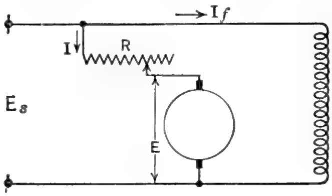


Fig. 170.

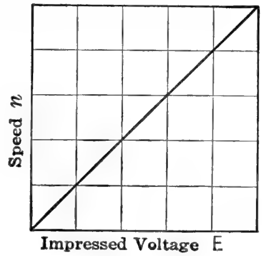


Fig. 171. Variation of speed with impressed voltage.

This method of varying speed is uneconomical as a large amount of power is lost in the control resistance; it is the product of the current input and the voltage consumed in the resistance and is

$$= I \times IR = I^2R \text{ watts.}$$

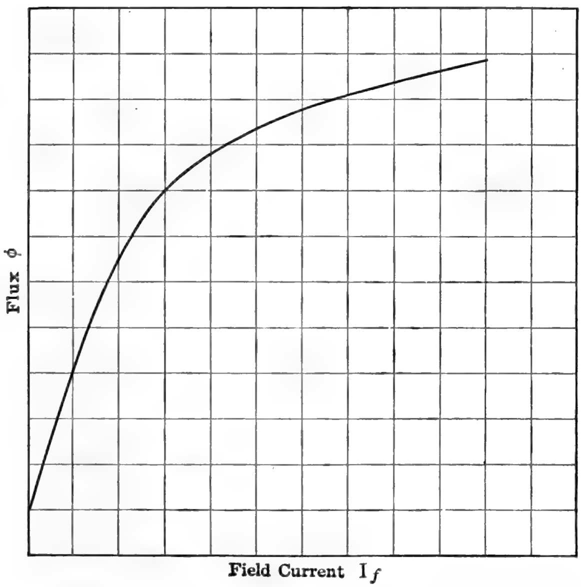


Fig. 172.

The resistance R must not be connected in series with the field winding as it would then decrease the field current and therefore the flux and tend to cause an increase in speed.

(2) When varying the speed by field control the full line voltage is impressed on the armature and a resistance R is connected in series with the field winding, Fig. 167. As the resistance R is increased the field current I_f decreases according to equation

$$I_f = \frac{E}{R + r_f},$$

and the flux Φ decreases with the field current as shown by the saturation curve of the machine in Fig. 172. Since the speed varies inversely as Φ, Fig. 173, the variation of speed with field current will be represented by a curve of the shape shown in Fig. 174.

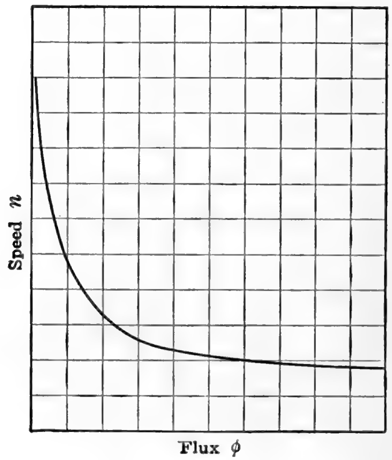


FIG. 173.

By this method the speed can be increased to any required value, and it tends to approach infinity when the field current is zero.

In machines of ordinary design the speed can be increased satisfactorily only about 70 per cent above normal speed by field weakening. Beyond this point it is not possible to get sparkless commutation of full-load current, since the armature m.m.f. is strong enough to overcome the weak field m.m.f. and wipe out the commutating field and at the same time the time of commutation is decreased.

By using interpoles which neutralize the effect of armature

m.m.f. and provide a commutating field, the speed can be increased to four times normal by field weakening without injurious sparking.

The loss in power in the resistance controlling the field current is very small since the power used for field excitation is only 1 or 2 per cent of the rated output.

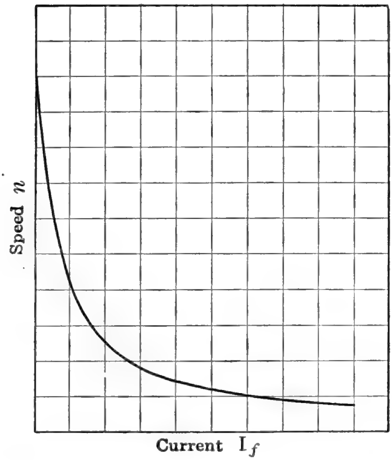


FIG. 174. Variation of speed with field current.

(3) At no load with the brushes on the neutral line, all the conductors on each half of the armature are effective in generating the back e.m.f. and this is therefore the position of minimum speed. (Fig. 126 (*a*).)

When the brushes are moved back against the direction of rotation, Fig. 126 (*b*), only the belts of conductors under the poles are effective in generating the back e.m.f. and the result is the same as though the flux had been decreased. The speed is therefore increased.

Under load, when current is flowing in the armature, the conductors between the poles exert a demagnetizing m.m.f. and cause a decrease in the flux; the speed therefore rises more than at no load.

This method of speed variation is not used to any extent since it interferes with commutation and causes injurious sparking.

125. Speed Characteristics of Motors. The speed characteristic is the curve showing the variation of speed with armature

current. The speed equation was found to be

$$n = \frac{E - Ir}{K\Phi} \text{ r.p.s.} \quad \text{(Art. 123.)}$$

Shunt Motor. As the motor is loaded and I increases the speed is affected in two ways, (1) the resistance drop Ir increases and tends to cause a proportionate decrease in speed; (2) the flux ϕ is decreased by the armature demagnetizing and cross-magnetizing m.m.f.; the demagnetizing effect increases directly with the current but the cross-magnetizing effect increases at a slower rate due to the saturation of the path; therefore the armature reaction tends to increase the speed. If the motor is operated at a point just above the knee of the saturation curve the drop in speed due to the armature resistance will be greater than the rise due to armature reaction and the speed characteristic will fall as shown in Fig. 175, curve 1.

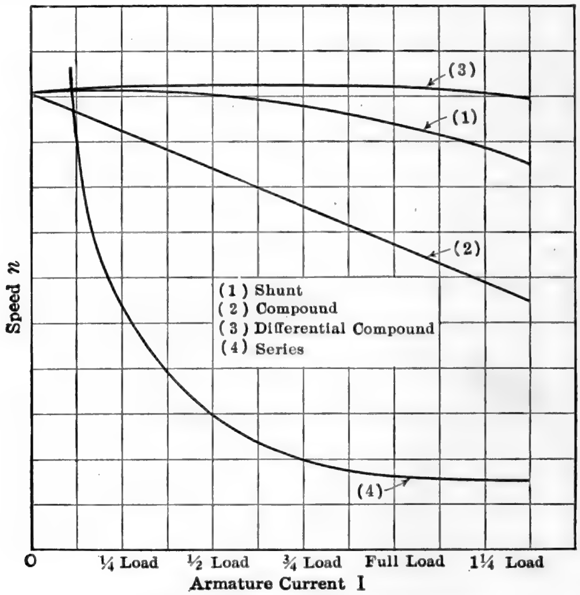


FIG. 175. Motor speed characteristics.

The speed regulation of a motor is the rise in speed when full load is thrown off expressed as a per cent of full-load speed.

The speed regulation of shunt motors of large size is from 2 to 5 per cent.

Compound Motor. In the compound-wound motor the armature resistance causes a drop in speed and the m.m.f. of the series winding overcomes the effect of the armature m.m.f. and increases the flux and thus decreases the speed more than in the shunt motor. A typical speed characteristic of a compound motor is shown in curve 2, Fig. 175.

If the series winding is reversed, its m.m.f. opposes the field m.m.f. and thus decreases the flux and causes the speed to increase with load as shown in curve 3, Fig. 175.

The motor is then called a "differential compound" motor and may be designed to give constant speed under all loads. If the series-field winding is strong the motor is unstable and tends to run at excessive speed.

Series Motor. At no load the series motor tends to run at a very high speed limited only by the residual magnetism or the torque required to overcome the losses. As load is applied the current and flux increase and the speed falls rapidly till the magnetic circuit of the machine becomes saturated; the speed characteristic then becomes almost horizontal. (Fig. 175, curve 4.)

126. Torque Equation. The torque of a motor is proportional to the product of the flux crossing the air gaps and the current in the armature. Its equation is derived as follows:

The e.m.f. impressed on the armature is

$$E = \mathcal{E} + Ir,$$

where

$$\mathcal{E} = Zn\Phi \frac{p}{p^1} 10^{-8}$$

is the back voltage generated in the armature and Ir is the voltage consumed by the armature resistance.

The power input to the armature is

$$EI = \mathcal{E}I + I^2r \text{ watts.} \quad \ldots\ldots \quad (217)$$

The power lost in the armature is I^2r watts, and thus the electric power transformed into mechanical power is

$$\mathcal{E}I = Zn\Phi I \frac{p}{p_1} 10^{-8};$$

this is the power output in watts neglecting the friction losses.

The motor output in horse power is

$$\frac{\mathcal{E}I}{746} = \frac{Zn\Phi I \dfrac{p}{p_1} 10^{-8}}{746}.$$

If the torque developed is T ft. lbs., then the output in horse power is

$$\frac{2\pi nT}{550} = \frac{Zn\Phi I \dfrac{p}{p_1} 10^{-8}}{746}$$

and the torque is

$$T = \frac{550}{2\pi n} \cdot \frac{Zn\phi I \dfrac{p}{p_1} 10^{-8}}{746} = 0.1177\, Z\phi I \frac{p}{p_1} 10^{-8} \text{ ft. lbs.} \qquad (218)$$

This is the torque equation of a direct-current motor.

$$T = 0.1177 \times 10^{-8} \times \left(Z \frac{I}{p_1}\right) \times (p\Phi) \text{ ft. lbs.,} \qquad (219)$$

where $Z \dfrac{I}{p_1}$ is the sum of all the currents in the armature conductors and $p\Phi$ is the sum of the fluxes crossing the air gaps under all the poles.

The torque equation may be written

$$T = K\Phi I \text{ ft. lbs.,} \quad \ldots \ldots \ldots \quad (220)$$

where $K = 0.1177\, Z \dfrac{p}{p_1} 10^{-8}$ is a constant.

The torque of a motor is therefore directly proportional to the armature current and to the flux in the air gap.

This is the torque developed at the shaft of the motor. The torque output is less due to the iron and friction losses.

127. Torque Characteristics of Motors. The torque characteristic is a curve showing the relation between the torque and the armature current.

The torque equation is

$$T = K\Phi I. \quad \text{(Art. 126.)}$$

Shunt Motor. In the shunt motor ϕ is almost constant under load, decreasing only by a small percentage due to armature reaction. The torque therefore varies almost directly with the current. (Curve 1, Fig. 176.)

Series Motor. In the series motor the flux increases almost in direct proportion to the current, while the magnetic circuit is unsaturated, and therefore the torque is proportional to the square of the current; at heavy load, when the magnetic circuit becomes saturated, the flux becomes almost constant and the torque then increases in direct proportion to the current. Curve 2, Fig. 176, is a typical torque-current curve for a series motor.

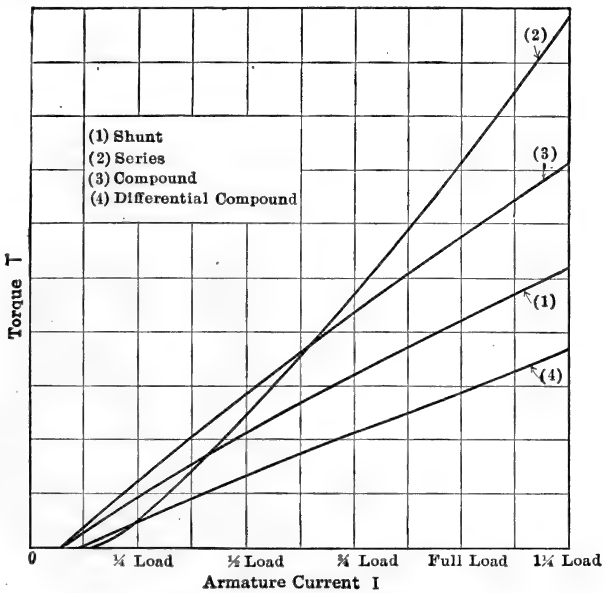


FIG. 176. Motor torque characteristics.

Compound Motor. In the compound motor the flux increases with load but not in direct proportion to the current; thus, the torque-current curve (curve 3) lies between those of the shunt and the series motor.

If the series winding is reversed, as in the differential compound motor, the flux decreases with load and the torque-current curve falls below that of the shunt motor. (Curve 4.)

128. Starting of Motors. Fig. 177 shows the proper connection for starting a shunt motor. Two conditions must be fulfilled. First, the field current must be as large as possible and therefore there must be no resistance in series with the field winding. This gives a large flux and therefore the required starting

torque can be obtained without excessive current in the armature. Second, resistance must be connected in series with the armature to limit the current and this resistance must be cut out gradually as the motor speeds up and generates a back voltage. If the resistance is cut out too quickly the motor draws a very large current and accelerates too rapidly. When all the resistance is out

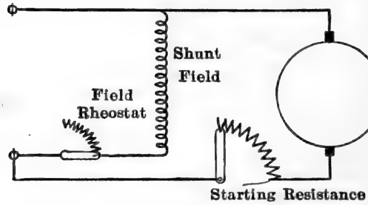


FIG. 177. Starting a shunt motor.

the motor runs at full speed and generates a back voltage almost equal to the applied voltage. If full voltage is applied to the armature at rest the current is limited by the armature resistance only, which is very small. Assume that the armature resistance of a 110-volt motor is 0.01 ohm. If 110 volts is applied the current tends to rise to a value $I = \frac{110}{0.01} = 11{,}000$ amperes and will immediately operate any protective devices on the system.

129. Applications of Motors. There are three types of direct-current motors, (1) shunt, (2) series and (3) compound.

(1) The characteristics of a shunt motor are (a) constant speed and (b) torque proportional to current.

Shunt motors are used for lathes, boring mills and all constant-speed machine tools, for driving line shafting when the starting load is not too heavy, for fans, centrifugal pumps, etc.

The starting torque is not large and when the motor has to start under load it draws an excessive current.

The speed of shunt motors can be increased by field weakening about 70 per cent above normal without injurious sparking if the output remains constant, or 100 per cent if the output is decreased to 80 per cent of full load.

For lathes speed ranges of 4 : 1 are necessary since the cutting speed must remain approximately constant although the diameter of the material changes. Field control must be used and in order to get the required range a multiple-wire supply system is often

employed. Fig. 178 shows such a system. Two generators, one giving 90 volts and the other 160 volts, are connected in series and three wires are run from them to the machine shop. The motor field is connected across 250 volts from a to b and a rheostat is placed in series with it. For low speeds the armature is connected across 90 volts from a to c and an increase in speed of 70 per cent can be obtained by the field rheostat. For medium

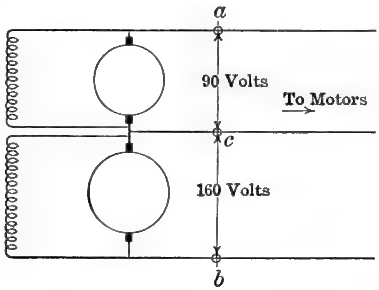


FIG. 178. Three-wire system.

speeds the armature is connected across 160 volts from c to b and the speed can again be increased 70 per cent. Finally for high speeds the armature is connected across 250 volts from a to b and the speed can be increased again by the field rheostat. In this way a range of 4 : 1 can easily be obtained.

With interpoles a speed range of 4 : 1 can be obtained by field weakening without using a three-wire system.

(2) The characteristics of a series motor are (a) variable speed and (b) torque proportional to the square of the current below saturation.

When a load comes on a series motor it responds by decreasing its speed and supplying the increased torque with a small increase of current, thus preventing a sudden shock on the supply system. A shunt motor under the same conditions would hold its speed nearly constant and would supply the required torque with a large increase of current and would thus make a heavy demand on the system.

Series motors must not be used for belt drives or in any case where the load may be removed suddenly since they run at excessive speed at light load.

Series motors are used in railway work and for cranes, hoists, etc., where very large starting torque is necessary.

(3) The characteristics of a compound motor are (a) variable speed and (b) torque greater than in the shunt motor but less than in the series motor.

At light loads compound motors approach a limiting speed which is fixed by their shunt excitation.

They are used for elevators and in classes of work where the load is variable and constant speed is not necessary and where a fairly large starting torque is required, except in those cases where series motors are necessary on account of their very large starting torque.

In rolling mills where the load fluctuates very rapidly a compound-wound motor is used with a heavy flywheel attached to it. When a heavy load comes on the speed falls and the flywheel gives up part of its energy. A similar motor with a flywheel is used to drive shears, punches, etc.

130. Power Losses in Dynamos. The power losses occurring in dynamos may be divided into copper losses, iron losses and friction losses, and these may again be subdivided as follows:

Copper losses.
- (a) Shunt-field copper loss.
- (b) Series-field copper loss.
- (c) Armature copper loss.

Iron losses.
- (d) Hysteresis loss.
- (e) Eddy current loss.

Friction losses.
- (f) Brush friction loss.
- (g) Journal friction loss.
- (h) Windage loss.

(a) The shunt-field copper loss is $I_f^2 r_f$ watts, where I_f is the current in the shunt-field winding and r_f is the resistance of the winding at the running temperature of the machine. This loss can be represented by $E_f I_f$, where $E_f = I_f r_f$ is the voltage impressed on the winding. Thus all the energy supplied to the field winding is transformed into heat and is wasted, since no energy is required to maintain the magnetic flux after it is once established.

The shunt-field loss is constant under all conditions of load and it ranges from 1 per cent of full-load output in large high-speed machines to 5 per cent in small low-speed machines.

If there is a rheostat connected in series with the shunt-field winding the power wasted in it should be included in the field copper loss.

(b) The series-field copper loss is $I_s^2r_s$ watts, where I_s is the current in the series winding and r_s is its resistance. This loss increases as the square of the load current of the machine. In interpole machines the resistance r_s will include the resistance of the interpole winding. The power loss in the shunts to the series winding or in the series-field rheostat must be included in the series-field loss.

(c) The armature copper loss may be divided into three parts, first, the loss due to the current I_a flowing through the resistance r_a of the armature winding, not including the resistance of the brush contacts. This part of the loss is $I_a^2r_a$ watts and increases as the square of the load current. Second, there is a loss of power where the current passes from the commutator to the brushes or vice versa. It is eI_a watts, where e is the drop of voltage at the brush contacts. The voltage e varies directly as the current at light loads when the current density is low but above a density of about 30 amperes per square inch of brush contact it remains nearly constant at a value of approximately 2 volts for ordinary carbon brushes. Above this point therefore the loss at the brush contacts increases as the first power of the current. This loss is negligible in high-voltage machines but is quite large in low-voltage machines. The third part of the armature copper loss is that caused by short-circuit currents in coils undergoing commutation or by circulating currents in the machine windings which may be produced by improper spacing of the brushes or by any variation in the depth of the air gaps under different poles in multipolar machines. These losses cannot be calculated.

(d) The hysteresis loss is due to the reversal of the magnetism in the armature iron as it moves across a pair of poles. Steinmetz found that the loss per cycle of magnetism varies as the 1.6th power of the induction density; it is

$$\omega_h = \eta \mathfrak{B}^{1.6} \text{ ergs},$$

where $\mathfrak{B}$ is the maximum induction in lines per square centimeter and η is the hysteretic constant for the iron and ω_η is the loss in ergs per cubic centimeter per cycle. The value of η for armature iron is about 0.003.

If the induction density B is expressed in lines per square inch and f is the number of cycles of magnetism per second, then the hysteresis loss per cubic inch of iron per second is

$$W_h = \eta\left(\frac{B}{(2.54)^2}\right)^{1.6}(2.54)^3 f \text{ ergs per second;}$$

but $$1 \text{ erg per second} = 10^{-7} \text{ watts,}$$

therefore, the loss in watts per cubic inch for a frequency of f cycles per second is

$$W_h = 0.83\,\eta B^{1.6} f\, 10^{-7} \text{ watts.} \quad . \quad . \quad . \quad . \quad (221)$$

In transformers where the induction density is nearly uniform throughout the volume of the iron this value multiplied by the volume of iron in cubic inches would give the hysteresis loss very closely; but, in the case of dynamos the induction density is not uniform, but varies from a maximum at the roots of the teeth to almost zero near the shaft, as shown in Fig. 179 and, therefore, the hysteresis loss cannot be calculated accurately but must be estimated from experience with similar machines.

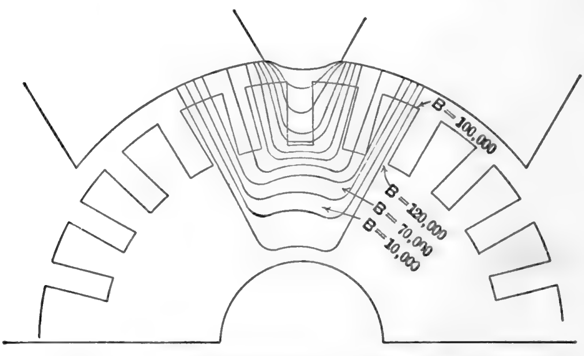


FIG. 179. Flux distribution in the armature core.

The hysteresis loss varies directly as the speed of the dynamo which is proportional to the frequency of the reversals of magnetism and it increases to a slight degree under load due to the distortion of the flux. In the regions where the density is increased the loss is increased more than it is decreased in the regions where the density is decreased.

(e) The eddy current loss is due to electric currents set up in

the armature iron by the e.m.f.'s generated in it as it cuts across the flux.

In Fig. 180 *abcd* represents a section of an armature punching of thickness t in. If the flux density in the gap is B lines per square

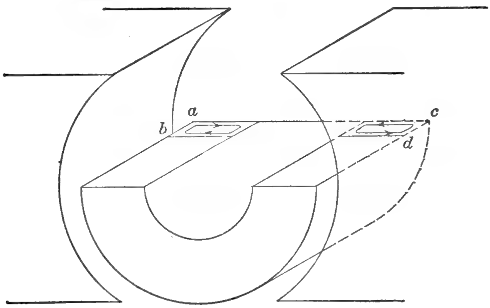


FIG. 180. Eddy current loss in the armature core.

inch and the edge *ab* is moving with a velocity of S ins. per second across the flux, then, the e.m.f. generated in the length *ab* is

$$e = BtS\,10^{-8} \text{ volts.}$$

This e.m.f. will cause a current to circulate through the iron as indicated by the arrow; the value of the current will be

$$i = \frac{e}{k\rho} = \frac{BtS\,10^{-8}}{k\rho} \text{ amperes,}$$

where ρ is the specific resistance of the iron and k is a constant depending on the dimensions of the section.

The loss in the section will be

$$p = i^2k\rho = \frac{B^2t^2S^2\,10^{-16}\,k\rho}{k^2\rho^2} = k_1\frac{B^2t^2S^2}{\rho} \text{ watts,} \quad . \quad (222)$$

where k_1, is a constant.

The eddy current loss, therefore, varies as the square of the induction density, the square of the thickness of the punchings and the square of the speed; it also depends on the specific resistance of the iron used but it cannot be calculated accurately in the case of a rotating armature, where the induction density varies throughout the section. It increases under load due to field distortion but tends to decrease as the temperature rises and increases the specific resistance of the iron.

Eddy currents are also produced in the pole faces due to local variations of the induction density as the armature teeth move

across them. This loss cannot be calculated but its presence is shown by the heating of solid pole faces. To reduce it the pole faces in direct-current machines should always be laminated.

(f) The brush friction loss in foot pounds per second is equal to the product of the total brush pressure in pounds, the peripheral speed of the commutator in feet per second and the coefficient of friction between the brush and the commutator. With carbon brushes the pressure should be from 1.5 to 2 lbs. per square inch; this value multiplied by the area of all the brushes gives the total brush pressure in pounds. In railway motors, where there is a great deal of vibration, pressures up to 5 lbs. per square inch are used in order to insure good contact. The coefficient of friction between a carbon brush and the commutator is about 0.3. The brush friction loss varies directly as the speed but is independent of the load.

(g) The journal friction loss increases as the (speed)$^{\frac{3}{2}}$ but is independent of the load.

(h) The windage loss depends on the shapes of the rotating parts of the machine and cannot be accurately determined. It varies almost as the cube of the speed but is usually very small.

The losses may be divided into two groups, the constant losses and the variable losses.

The constant losses are those which do not vary to any great extent under load and include the shunt-field copper loss, the iron losses and the friction and windage losses.

The variable losses are those which increase with load, namely, the armature copper loss and the series-field copper loss.

All the losses in a machine appear as heat and raise the temperature of the various parts.

131. Efficiency. The efficiency of a machine may be variously expressed as,

$$\begin{aligned}\eta &= \frac{\text{output}}{\text{input}}\,100 \text{ per cent} \\ &= \frac{\text{output}}{\text{output} + \text{losses}}\,100 \text{ per cent} \\ &= \frac{\text{input} - \text{losses}}{\text{input}}\,100 \text{ per cent.} \quad . \quad . \quad . \quad (223)\end{aligned}$$

The efficiency varies with the output; at light loads it is low on account of the constant losses; between $\frac{3}{4}$ load and full load it is

maximum and the constant losses and variable losses are nearly equal; above full load it decreases again due to the rapid increase of the variable losses. The limit of the efficiency which can be reached commercially depends on the output, the voltage and the speed. A higher efficiency can be obtained with large machines than with small machines. A higher efficiency can be obtained with high-voltage or high-speed machines than with low-voltage or low-speed machines.

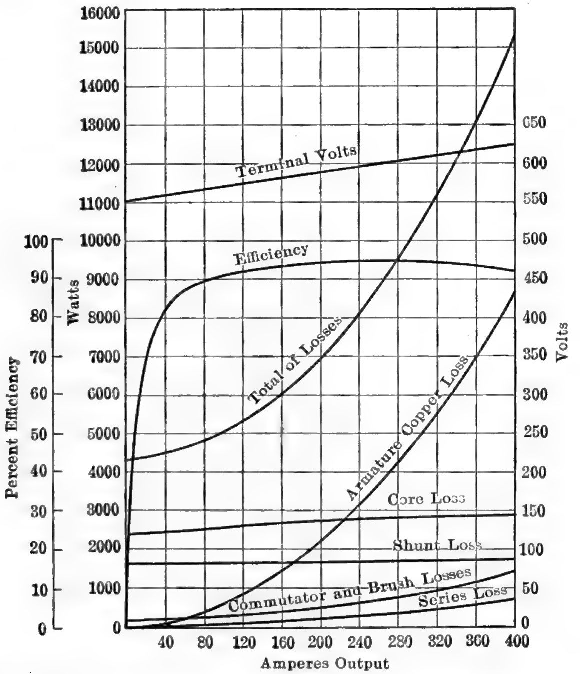


FIG. 181. Characteristic curves of a 200-kw. compound-wound generator.

For 220-volt direct-current motors the full-load efficiency ranges from about 85 per cent for small sizes to 93 per cent for large sizes.

For 550-volt direct-current generators the full-load efficiency ranges from about 90 per cent for small sizes to 95 per cent for large sizes.

Fig. 181 shows the characteristic curves of a 200-kw. compound-wound generator, 550 to 625 volts.

Fig. 182 shows the characteristic curves of a 75-h. p., 600-volt railway motor built by the General Electric Co.

Fig. 183 shows the characteristic curves of a 500-volt crane motor built by the Crocker Wheeler Co. Its rated output is 65 h. p. for $\frac{1}{2}$ hour with a temperature rise of 40° C.

Fig. 184 shows the characteristic curves of a 25 h. p., 500-volt compound-wound motor of the Westinghouse Co.

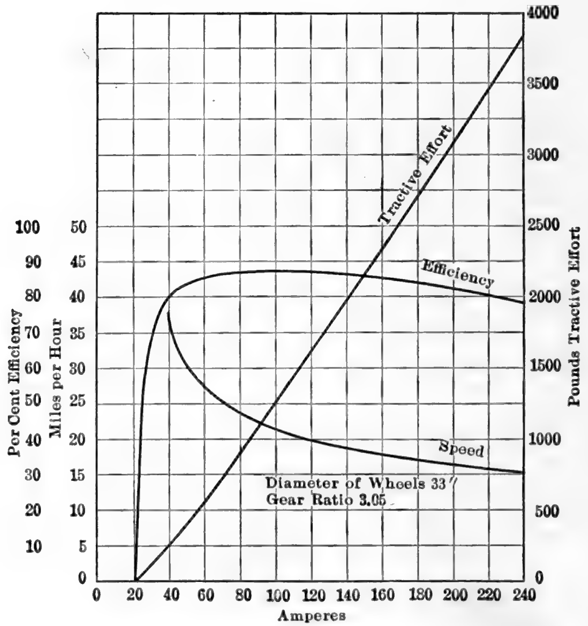


FIG. 182. Characteristic curves of a 75-h. p. railway motor.

132. Limits of Output of Electric Machines. The factors which limit the output of electric machines are

(1) regulation,
(2) efficiency,
(3) heating,
(4) sparking.

(1) In motors the regulation is a speed regulation. With increase of load the speed falls off and the increased torque is obtained at a decreased speed. In constant-speed work the shunt

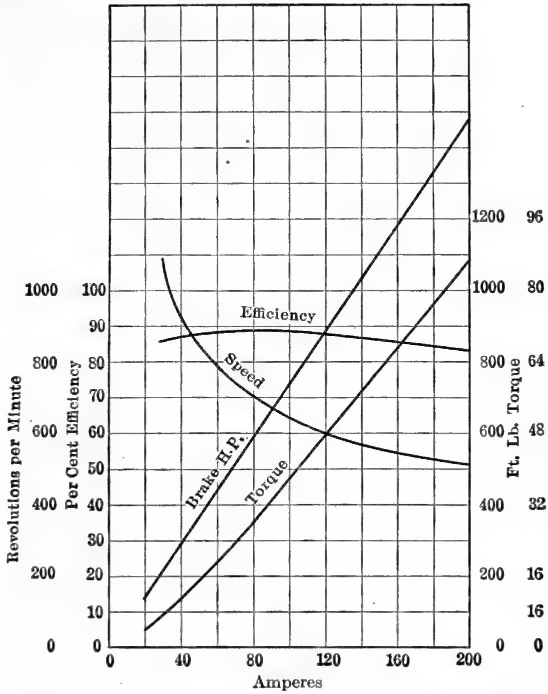


FIG. 183. Characteristic curves of a 500-volt crane motor with a capacity of 65 h. p. for ½ hour.

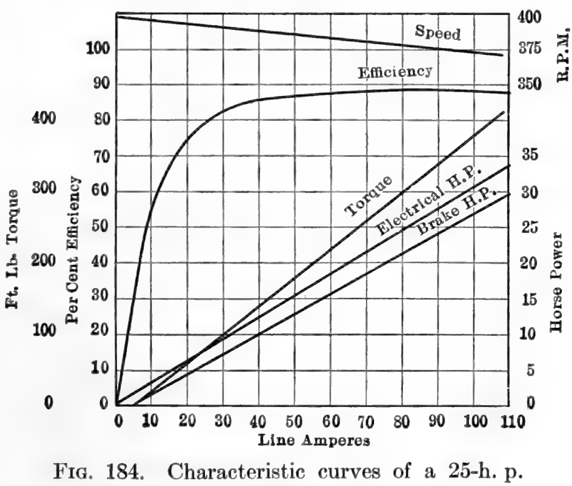


FIG. 184. Characteristic curves of a 25-h. p. compound-wound motor.

motor is used but if it is overloaded its speed falls to a value too low for satisfactory operation.

In generators the regulation is a voltage regulation. As the load is increased the voltage falls off and a point is soon reached where any increase of load will cause so great a decrease in voltage that the power supplied is unsatisfactory.

(2) The efficiency of a machine increases with increasing load to the point where the variable copper losses are equal to the constant losses. Above this point the efficiency decreases due to the rapid increase of the variable losses.

With properly designed machines the output is limited by either heating or sparking before the regulation or efficiency become too bad.

(3) All the losses of power in a machine are converted into heat and raise the temperature of the various parts until the point is reached where the rate at which heat is being radiated or carried off by the ventilating apparatus is equal to the rate at which heat is being generated. The temperature will then remain constant. When a machine is overloaded its losses increase and consequently its temperature rises above normal.

If a machine operates at a high temperature for any length of time permanent injury to the insulating materials will result.

(4) Sparking will occur in a machine when the field cut by the coil which is being commutated is not strong enough to reverse the current in the time of commutation. Sparking will therefore occur in generators or motors at heavy load when the armature m.m.f. is so great that it wipes out the field under the pole tip or weakens it to such an extent that it cannot produce the required commutating e.m.f. Motors will also spark at high speed since the time of commutation is reduced, especially when the high speed is produced by field weakening. Take for example a shunt motor rated at normal speed as 10 h. p., 110 volts, 80 amperes, 400 r. p. m. and suppose the temperature rise to be 50° C. above standard room temperature of 25° C. If the motor is operated at half speed of 200 r. p. m. by reducing the impressed voltage to half, the rating may be taken as 10 h. p., 55 volts, 80 amperes but the temperature rise will be greater than before because the armature copper loss is the same, the field copper loss is the same and the iron and friction losses are less due to the low speed, but the ventilation is only about half as good as before.

When operated at twice full speed produced by field weakening, the rating may be taken as 10 h. p., 110 volts, 80 amperes but the temperature rise will be less than before, because the armature copper loss is the same, the field copper loss is reduced to about one quarter of its normal value and the iron and friction losses are increased but the ventilation is very much improved. The rated output of the machine for normal temperature rise might be increased but due to the higher speed and consequent reduced time of commutation sparking would occur.

133. Storage Batteries. A storage battery is an apparatus in which electrical energy can be stored to be used at some later time.

Batteries are made up of a number of cells connected in series multiple according to the voltage and current required.

Each cell is composed of two plates or electrodes of suitable materials immersed in an electrolyte. The most commonly used storage battery has a positive plate of lead peroxide PbO_2 and a negative plate of sponge lead Pb immersed in dilute sulphuric acid H_2SO_4.

When the battery is discharging the electrolyte combines with the active materials of the electrodes and when it is being charged the electrodes are reduced to their original condition and the materials taken from the electrolyte are returned to it.

The main chemical changes taking place are represented by the following formula:

$$PbO_2 + \underset{\xrightarrow[\text{discharge}]{}}{\overset{\xleftarrow{\text{charge}}}{Pb + 2\,H_2SO_4 = 2\,PbSO_4}} + 2\,H_2O. \quad . \quad (224)$$

Capacity. The unit of capacity of a storage cell is the ampere hour and it is generally based on the eight-hour discharge rate. An 800-ampere-hour battery will give a continuous discharge of 100 amperes for eight hours. If, however, the rate of discharge is increased the ampere-hour capacity of the battery decreases. At a six-hour discharge rate the capacity is only about 95 per cent, at a four discharge rate it is about 80 per cent and at a one-hour rate it is only 50 per cent of its eight-hour rating. Thus the battery mentioned above would give a continuous discharge of 400 amperes for only one hour.

The capacity of a cell is proportional to the area of the plates exposed to the electrolyte and for an eight-hour discharge rate a current density of from 40 to 60 amperes per square foot of positive plate is common practice.

Voltage. The voltage of a cell depends on the character of the electrodes, the density of the electrolyte and the condition of the cell but is independent of the size. The variation of the terminal voltage of a cell during charge and discharge is shown by the curves in Fig. 185. On charge the voltage begins about 2 volts

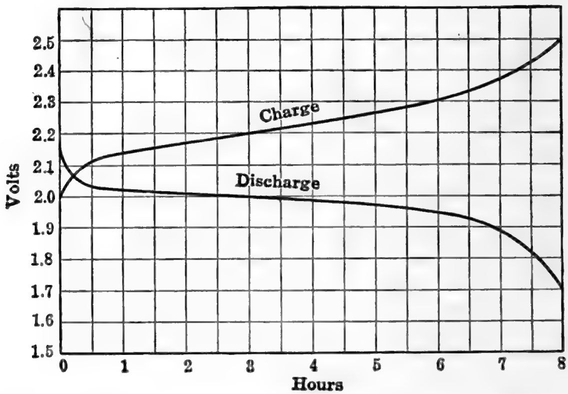


FIG. 185. Voltage characteristics of a storage cell.

and rises to 2.5 volts or a little above. When the charging circuit is opened the voltage falls to 2.1 volts and during discharge falls off gradually to about 1.9 volts. Beyond this point the fall of voltage is very rapid and discharge should not be continued after the voltage has fallen to 1.7 volts.

The required battery voltage is obtained by connecting a number of cells in series and the required current is obtained by connecting a number of plates or cells in multiple.

134. Applications. Batteries are installed in direct-current power stations to store energy during periods of light load and to deliver energy in parallel with the generators during periods of heavy load. When the load is light the generators charge the battery and when the load is heavy the charge is given up and so the load on the generators is maintained nearly constant and they can be operated at maximum efficiency. The result is that the voltage regulation of the system is improved.

Batteries are also installed in electric railway substations to prevent large variations of the load on the feeders supplying them and so regulate the substation voltage.

A third very important application of storage batteries is in alternating-current power stations where they provide an auxiliary supply of direct current in case of a breakdown of the exciters and may thus prevent a shutdown of the whole system.

Fig. 186 shows a battery connected across the terminals of a shunt generator. At normal load the battery voltage and the generator voltage are equal and the battery floats on the line neither giving nor receiving power. If, however, the load increases the generator voltage falls and the battery discharges and supplies

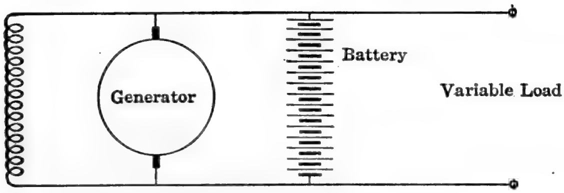


FIG. 186. Battery with shunt generator.

part of the extra load and so relieves the generator and prevents any large drop in its terminal voltage. The battery in this way takes care both of sudden overloads and continuous overloads. During periods of light load the generator voltage rises above the battery voltage and the battery charges.

135. Boosters. Boosters are direct-current generators connected in series with the line to raise or lower the voltage; they are used very extensively to regulate the charge and discharge of storage batteries without changing the generator voltage. They may be either shunt, series or compound wound.

The shunt booster, Fig. 187, is an ordinary generator with its field connected across the station bus bars and its armature connected in series with the main generator. Its function is to raise the voltage impressed on the battery in order to send current into it to charge it. The voltage of the booster is controlled by a field rheostat.

Compound boosters are automatic in their action and are divided into two classes, non-reversible and reversible, depending on the relative strengths of their shunt and series windings.

Fig. 188 shows a non-reversible automatic booster. The shunt field f is connected across the station bus bars. The series field s carries the load current of the generator and it opposes the shunt field but has at all times a smaller m.m.f. and thus the booster voltage is always in the direction of the generator voltage.

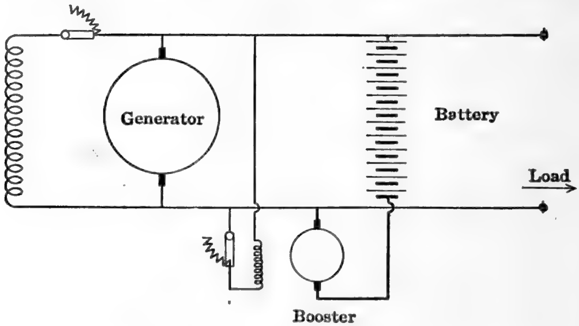


FIG. 187. Shunt booster.

When the load current increases an increase of current through s decreases the booster voltage and allows the battery to discharge; when the load decreases the booster voltage rises and causes the battery to charge. The current from the generator remains practically constant regardless of the fluctuations of load.

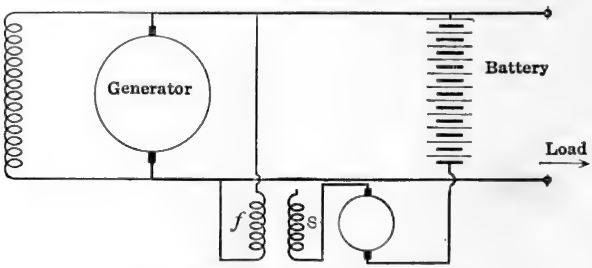


FIG. 188. Compound booster.

The reversible booster is similar in construction to the non-reversible booster but has a stronger series field. At normal load the shunt and series fields are of equal strength and the booster voltage is zero. The battery then floats on the line and neither charges nor discharges. An increase of load above normal increases the strength of s and overpowers f and so discharges the battery. When the load decreases f becomes stronger than s and causes the battery to charge.

Thus by installing an automatic booster the battery is made more sensitive to variations of load and a better regulation of the generator load and voltage is obtained.

Series boosters are used in electric railway engineering and in general power distribution to raise the voltage on certain sections

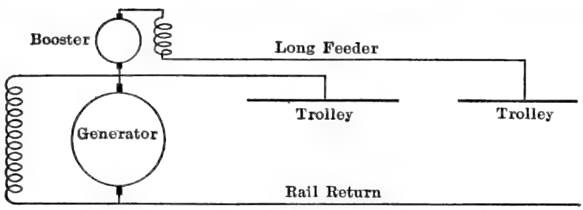


FIG. 189. Series booster.

of the line as in the case of a long feeder supplying power to an outlying section, as shown in Fig. 189. The booster field is connected in series with the line and it produces an increase of the impressed voltage to take care of the line drop.

CHAPTER V

SYNCHRONOUS MACHINERY

136. Alternator. An alternator consists essentially of an open coil of wire revolving at uniform speed in the magnetic field between a pair of unlike poles. (Fig. 190.) The field m.m.f. is produced by windings excited by direct current.

Between the slip rings a and b an alternating e.m.f. is generated of instantaneous value

$$e = n \frac{d\phi}{dt} 10^{-8} \text{ volts}, \quad . \quad . \quad . \quad . \quad . \quad (225)$$

where n is the number of turns in the coil and $\frac{d\phi}{dt}$ is the rate of change of the flux interlinking with the coil or the rate at which the coil is cutting the flux. The result is the same if the coil is stationary and the field revolves.

137. Types of Alternators. There are three principal types of alternators,

(a) revolving armature,
(b) revolving field,
(c) inductor.

Type (a) is illustrated in Fig. 190. The field poles are stationary and the armature revolves between them. The ends of the winding are brought out to two slip rings in single-phase machines and to three or more slip rings in polyphase machines and the current is collected by copper or carbon brushes.

The armature is necessarily of small size since the peripheral speed is limited and there is very little space for insulation. The armature conductors are also acted upon by centrifugal forces which tend to throw them out of the slots. The revolving armature is therefore only suitable for machines of small size and low voltage. It is, however, necessary in the case of rotary converters where the same armature winding carries both alternating and direct currents.

(b) The revolving field type illustrated in Fig. 191 and Fig. 192, is almost universal for all sizes and voltages. The armature is the stationary part and the field poles revolve. This type has many advantages over the revolving armature type. (1) It only requires two slip rings even for polyphase machines and these

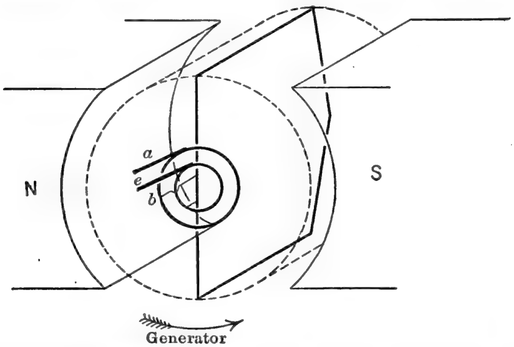


FIG. 190. Single-phase alternator, revolving armature type.

slip rings carry only the small current supplied to the field winding, while the load current is taken off from stationary terminals. (2) There is much more space for the armature windings and they are relieved from all centrifugal strains. They can, therefore, be much better insulated and ventilated. The field windings are made of copper strap and the revolving member is very

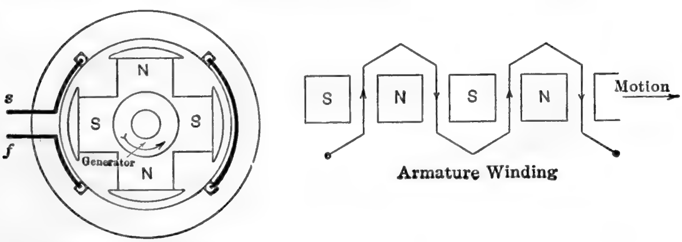


FIG. 191. Single-phase alternator, revolving field type.

rugged and is not affected by strains due to rotation; thus, much higher peripheral speeds may be used than with type (a), with consequent increase in economy of material.

(c) The inductor alternator is almost obsolete. In these machines the field and armature windings are both stationary and a part of the iron of the magnetic circuit revolves, producing

a periodic pulsation of the reluctance of the magnetic circuit and consequently a variation of the flux linking with the armature winding. Fig. 193 represents one type of inductor alternator.

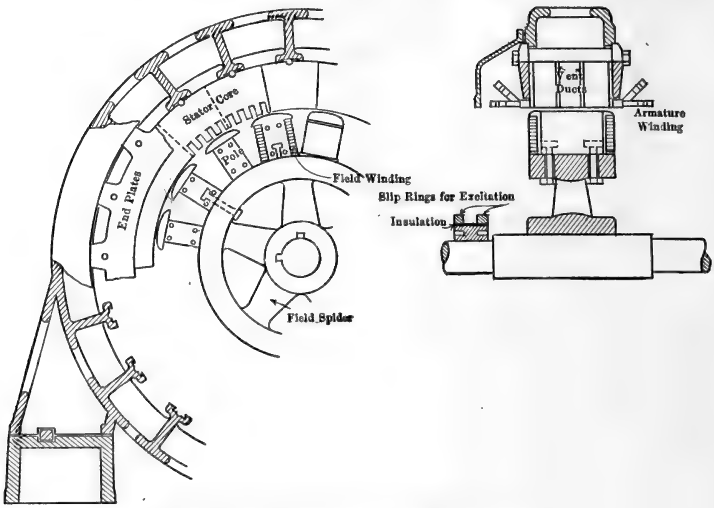


FIG. 192. Revolving field alternator.

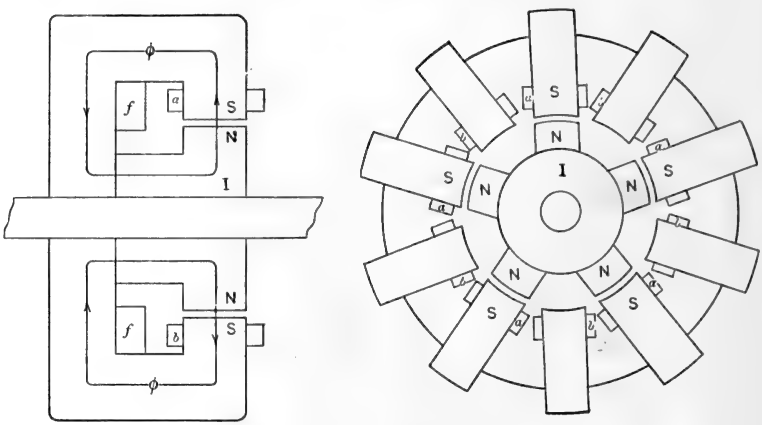


FIG. 193. Inductor alternator.

ff is the stationary field winding which produces the magnetic flux ϕ,ϕ, in the direction indicated. a,b, a,b, are the armature coils which may be connected either in series or in multiple. *I*

is the revolving part of the magnetic circuit and is called the inductor. The polar projections on it are all north poles but the amount of flux issuing from I and linking with the armature coils a,a, b,b, depends on the relative position of the inductor projections and the projections carrying the armature coils.

In Fig. 194, curve 1 represents the variation of the flux ϕ_a interlinking with one armature coil a starting from the position when this flux is maximum, Fig. 193. Curve 2 represents the e.m.f. generated in coil a by the varying flux. As ϕ_a decreases

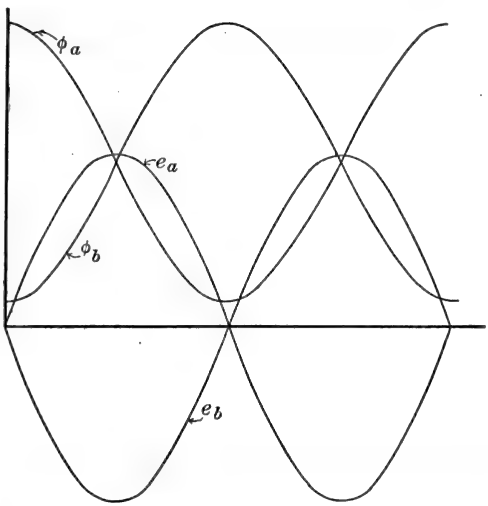


FIG. 194. Fluxes and e.m.f.'s in an inductor alternator.

an e.m.f. is generated in the positive direction and as it increases again an e.m.f. is generated in the negative direction. Thus although the flux does not reverse its direction and never reaches zero, an alternating e.m.f. is generated in the armature coil. If the pole pieces are properly shaped a sine wave of e.m.f. will be produced. Curves 3 and 4 represent the variation of flux and e.m.f. in coil b. The e.m.f. in coil b is displaced 180 degrees from that in a and before the two are connected in series the terminals of one must be reversed.

Inductor alternators were very heavy and expensive and have been superseded by the other types.

Any one of these three types may be wound as single-phase or polyphase machines.

138. Electromotive Force Equation. Fig. 190 represents a two-pole, single-phase alternator. The armature winding is a single coil of n turns revolving at a constant speed of N r.p.s. The magnetic field is assumed to be uniform.

The e.m.f. generated in the winding goes through one complete cycle during each revolution, and thus the frequency in cycles per second is equal to the speed in revolutions per second or $f = N$.

The angular velocity of the coil is ω radians per second, and therefore

$$\omega = 2\pi N = 2\pi f. \qquad (226)$$

If Φ is the maximum flux inclosed by the coil, that is, the flux inclosed when the coil is vertical, as shown, and time is measured from this instant, then at time t, after the coil has turned through an angle θ, the flux inclosed is $\phi = \Phi \cos\theta$, and the e.m.f. generated in the coil is

$$e = -n\frac{d\phi}{dt}10^{-8}$$

$$= -n\frac{d}{dt}(\Phi\cos\theta)\,10^{-8},$$

but $\theta = \omega t = 2\pi ft$, and thus

$$e = -n\frac{d}{dt}(\Phi\cos 2\pi ft)\,10^{-8}$$

$$= 2\pi fn\Phi\,10^{-8}\sin 2\pi ft \text{ volts} \qquad (227)$$

$$= E_0 \sin\theta.$$

This is a sine wave of maximum value

$$E_0 = 2\pi fn\Phi\,10^{-8} \text{ volts} \qquad (228)$$

and effective value

$$E = \frac{E_0}{\sqrt{2}} = 4.44\,fn\Phi\,10^{-8} \text{ volts.} \qquad (229)$$

This is the electromotive force equation for an alternator which produces a sine wave of e.m.f. and has a concentrated winding, that is, all the turns wound in a single coil.

This result may also be obtained as follows. The flux cut per second by each turn of the coil is $4f\Phi$ lines, and therefore the average e.m.f. generated in the coil is

$$E_{avg} = 4\,fn\Phi\,10^{-8} \text{ volts,} \qquad (230)$$

but for a sine wave the ratio of the maximum to the average ordinate is $\frac{\pi}{2}$, and therefore the maximum e.m.f. is

$$E_0 = \frac{\pi}{2} E_{avg} = 2\,\pi f n \Phi \, 10^{-8},$$

and the effective value is as before

$$E = \frac{E_0}{\sqrt{2}} = 4.44\, f n \Phi \, 10^{-8}.$$

The e.m.f. generated in an alternator is directly proportional to the frequency f, to the number of turns in series n and to the flux under each pole Φ.

In a two-pole machine one revolution or 360 mechanical degrees corresponds to one cycle or 360 electrical degrees and the frequency is equal to the number of revolutions per second; in a p-pole machine the e.m.f. goes through a complete cycle when the coil moves across a pair of poles and thus through $\frac{p}{2}$ cycles in one revolution. In this case 360 mechanical degrees $= \frac{p}{2} \times 360$ electrical degrees or one mechanical degree $= \frac{p}{2}$ electrical degrees.

The frequency in a p-pole alternator revolving at N r.p.s. is

$$f = \frac{p}{2} N \text{ cycles per second.} \quad . \quad . \quad . \quad . \quad . \quad (231)$$

139. Form Factor. If the flux in the air gap is not so distributed as to give a sine wave of e.m.f., the average value of the generated e.m.f. is still given by equation 230,

$$E_{avg} = 4\, f n \Phi \, 10^{-8},$$

but the ratio of maximum to average value is not $\frac{\pi}{2}$ and the ratio of effective to maximum value is not $\frac{1}{\sqrt{2}}$.

The effective value of the general alternating wave can be expressed as

$$E = 4\, \gamma f n \Phi \, 10^{-8} \text{ volts,} \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

where γ is called the form factor of the wave and is defined as the ratio of the effective value to the average value of the ordinate of the wave.

The form factor of a rectangular wave is 1.00 and for all other waves is greater than 1.00. For a sine wave the form factor is

$$\gamma = \frac{E_{eff}}{E_{avg}} = \frac{\frac{E_0}{\sqrt{2}}}{\frac{E_0}{\frac{\pi}{2}}} = \frac{\pi}{2\sqrt{2}} = 1.11.$$

140. Two-phase or Quarter-phase Alternator. When two similar single-phase windings are placed on an armature spaced at 90 electrical degrees to one another the winding is called a two-phase winding and two e.m.f.'s are generated displaced in phase by 90 degrees. Fig. 195 shows a two-phase winding.

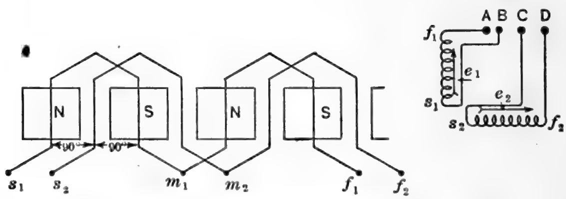


FIG. 195. Two-phase alternator winding.

Phase 1 starts at s_1 and ends at f_1 and phase 2 starts at s_2 and ends at f_2.

The e.m.f. generated in phase 1 is

$$e_1 = E_0 \sin \theta, \text{ effective value } E = \frac{E_0}{\sqrt{2}};$$

the e.m.f. generated in phase 2 is

$$e_2 = E_0 \sin (\theta - 90), \text{ effective value } E = \frac{E_0}{\sqrt{2}};$$

and they are represented by the two waves or the two vectors in Fig. 196.

The two windings are usually entirely separate and their ends are brought out to four terminals A, B, C, D.

If any two terminals as B and C are joined the e.m.f. between A and D is

$$\begin{aligned} e_{AD} &= e_1 - e_2 = E_0 \sin \theta - E_0 \sin (\theta - 90) \\ &= E_0 (\sin \theta + \cos \theta) \\ &= \sqrt{2}\, E_0 \sin (\theta + 45), \qquad (233) \end{aligned}$$

it leads e_1 in phase by 45 degrees, its maximum value is $\sqrt{2}\,E_0$ and its effective value is $E_{AD} = \dfrac{\sqrt{2}\,E_0}{\sqrt{2}}; = E_0 = \sqrt{2}\,E.$ This value can also be obtained by subtracting the two vectors as shown in Fig. 197.

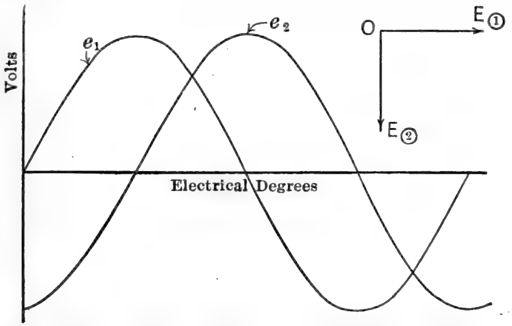


FIG. 196. E.m.f. waves of a two-phase alternator.

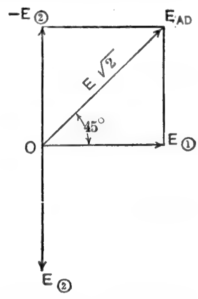


FIG. 197.

The middle points m_1 and m_2 of the two windings are sometimes connected together and a fifth terminal F used as shown in Fig. 198. The common terminal F is called the neutral point of the winding and may be connected to earth. The e.m.f. from each of the other terminals to F is the same.

$$E_{AF} = E_{BF} = E_{CF} = E_{DF} = \frac{E}{2}.$$

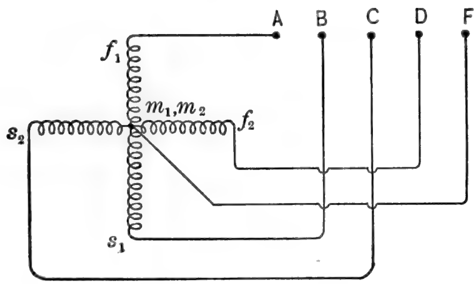


FIG. 198.

The four e.m.f.'s E_{AC}, E_{CB}, E_{BD} and E_{DA} are equal, since each is the vector difference of two e.m.f.'s of effective value $\frac{E}{2}$ at right angles to one another, and these four e.m.f.'s are also at right angles to one another and form a four-phase or quarter-phase system.

The effective value of each of the four e.m.f.'s is

$$\sqrt{\left(\frac{E}{2}\right)^2 + \left(\frac{E}{2}\right)^2} = \frac{E}{\sqrt{2}}.$$

141. Three-phase Alternator. If three similar windings are placed on the same alternator armature displaced 120 electrical degrees from one another, Fig. 199, and the ends of the windings

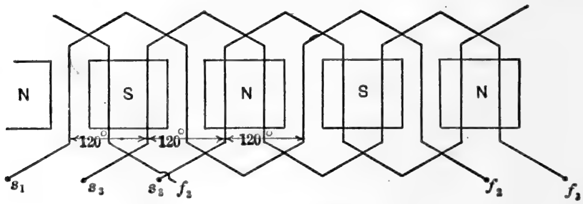


FIG. 199. Three-phase alternator winding.

are brought out to terminals, the machine is a three-phase alternator. The e.m.f.'s generated in the three windings are displaced 120 degrees.

The e.m.f. in phase 1 is $e_1 = E_0 \sin \theta$, effective value $E = \frac{E_0}{\sqrt{2}}$;

the e.m.f. in phase 2 is $e_2 = E_0 \sin (\theta - 120)$, effective value $E = \frac{E_0}{\sqrt{2}}$;

and the e.m.f. in phase 3 is $e_3 = E_0 \sin (\theta - 240)$, effective value $E = \frac{E_0}{\sqrt{2}}$.

The windings may be interconnected in two ways. (1) Join f_1 to s_2, f_2 to s_3 and f_3 to s_1, and connect the three junctions to

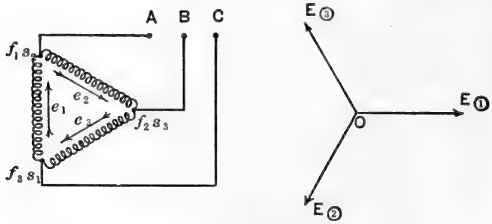


FIG. 200. Delta connection.

the terminals A, B and C. (Fig. 200.) This is called the "delta" connection or ring connection and is represented by Δ.

The resultant e.m.f. around the closed circuit at any instant is

$$\begin{aligned} e_1 + e_2 + e_3 &= E_0 \sin\theta + E_0 \sin(\theta - 120) + E_0 \sin(\theta - 240) \\ &= E_0 (\sin\theta + \sin\theta \cos 120 - \cos\theta \sin 120 \\ &\qquad + \sin\theta \cos 240 - \cos\theta \sin 240) \\ &= E_0 \left(\sin\theta - \tfrac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta - \tfrac{1}{2}\sin\theta \right. \\ &\qquad \left. + \frac{\sqrt{3}}{2}\cos\theta\right) = 0, \end{aligned}$$

and therefore if the three windings are exactly similar and are displaced exactly 120 degrees there will be no resultant e.m.f. acting around the winding and no circulating current will flow. This result has been derived on the assumption that the e.m.f. wave is a true sine wave. If the e.m.f. wave is made up of a fundamental and a third harmonic, the third harmonics in the three phases will be displaced by $3 \times 120° = 360°$ and will therefore be in phase and will combine to a resultant third harmonic of three times the magnitude of that in one phase and this will cause a third harmonic of current to circulate through the closed winding. This current may be of the order of full-load current in the case of alternators of low reactance. The reactance of the alternator winding to the triple frequency current is three times that offered to the current of fundamental frequency but only the true reactance and not the synchronous reactance of the winding is effective in limiting the circulating current. The third harmonic of e.m.f. does not appear at the terminals since it is consumed in producing the circulating current.

(2) If the ends s_1, s_2 and s_3 are connected together and the ends f_1, f_2 and f_3 are joined to the three terminals A, B and C, Fig. 201, the windings are connected "Y" or "star."

The e.m.f. between A and B is

$$\begin{aligned} e_{AB} = e_1 - e_2 &= E_0 \sin\theta - E_0 \sin(\theta - 120) \\ &= E_0 (\sin\theta - \sin\theta \cdot \cos 120° + \cos\theta \cdot \sin 120) \\ &= E_0 \left(\sin\theta + \tfrac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) \\ &= \sqrt{3}\, E_0 \left(\frac{\sqrt{3}}{2}\sin + \tfrac{1}{2}\cos\theta\right) \\ &= \sqrt{3}\, E_0 \sin(\theta + 30); \quad \ldots \ldots \quad (234) \end{aligned}$$

its maximum value is $\sqrt{3}\,E_0$ and its effective value is

$$\sqrt{3}\frac{E_0}{\sqrt{2}} = \sqrt{3}\,E.$$

Similarly the e.m.f. between B and C is

$$\begin{aligned} e_{BC} &= e_2 - e_3 = E_0\{\sin(\theta - 120) - \sin(\theta - 240)\} \\ &= \sqrt{3}\,E_0 \sin(\theta - 90) \quad \text{. . . . . (235)} \end{aligned}$$

of maximum value $\sqrt{3}\,E_0$ and effective value $\sqrt{3}\,E$.

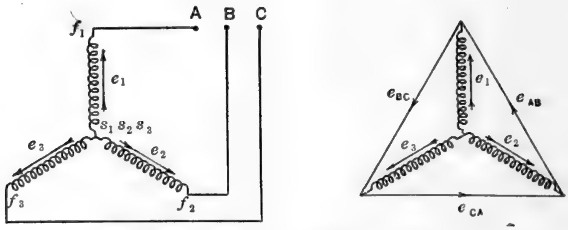


FIG. 201. Star or "Y" connection.

The e.m.f. between C and A is

$$\begin{aligned} e_{CA} &= e_3 - e_1 = E_0\{\sin(\theta - 240) - \sin\theta\} \\ &= \sqrt{3}\,E_0 \sin(\theta + 150) \quad \text{. . . . . (236)} \end{aligned}$$

of maximum value $\sqrt{3}\,E_0$ and effective value $\sqrt{3}\,E$.

Thus the three e.m.f.'s between terminals are equal to one another and are displaced in phase by 120 degrees.

A fourth terminal is usually connected to the neutral point 0 and it may be grounded.

If a third harmonic exists in the e.m.f. wave of each phase it will not appear in the e.m.f. between terminals in the "Y" con-

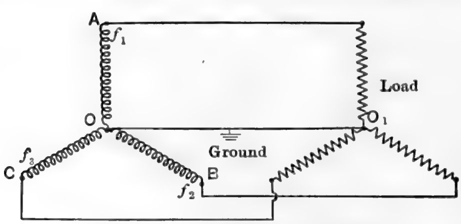


FIG. 202. Grounded neutrals.

nection since this e.m.f. is the difference of two e.m.f.'s at 120 degrees to one another or the sum of two e.m.f.'s at 60 degrees, the third harmonics will be combined at $3 \times 60° = 180°$ and will

therefore neutralize one another. If, however, the neutral is connected to ground at both the generator and receiver end, as shown in Fig. 202, a third harmonic of current may flow in the neutral supplied from the three phases.

Alternators should, whenever possible, be connected in "Y" instead of "Δ" to reduce the danger of circulating currents.

142. E.M.F.'s, Currents and Power in Three-phase Circuits. Fig. 203 shows a "Δ"-connected three-phase system.

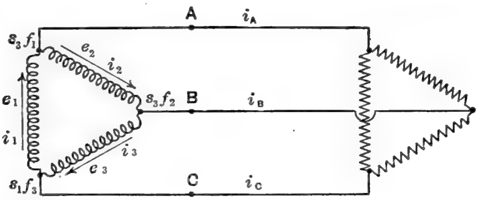


FIG. 203.

The e.m.f. in phase 1 is

$$e_1 = E_0 \sin \theta, \text{ effective value } E_1 = \frac{E_0}{\sqrt{2}};$$

the e.m.f. in phase 2 is

$$e_2 = E_0 \sin (\theta - 120), \text{ effective value } E_2 = \frac{E_0}{\sqrt{2}};$$

the e.m.f. in phase 3 is

$$e_3 = E_0 \sin (\theta - 240), \text{ effective value } E_3 = \frac{E_0}{\sqrt{2}}.$$

The effective values of the e.m.f.'s in the three phases are equal and are the terminal e.m.fs. of the alternator or the e.m.fs. between lines,

$$E_t = E_1 = E_2 = E_3 = \frac{E_0}{\sqrt{2}}.$$

The current in phase 1 is

$$i_1 = I_0 \sin (\theta - \phi), \text{ effective value } I_1 = \frac{I_0}{\sqrt{2}};$$

the current in phase 2 is

$$i_2 = I_0 \sin (\theta - \phi - 120), \text{ effective value } I_2 = \frac{I_0}{\sqrt{2}};$$

the current in phase 3 is

$$i_3 = I_0 \sin (\theta - \phi - 240), \text{ effective value } I_3 = \frac{I_0}{\sqrt{2}}.$$

The effective value of the currents in the three phases is the same and is $I = \frac{I_0}{\sqrt{2}}$.

ϕ is the angle of lag of the current in each phase behind the e.m.f. generated in that phase.

The current in line A is

$$i_A = i_1 - i_2 = I_0 \sin(\theta - \phi) - I_0 \sin(\theta - \phi - 120)$$
$$= \sqrt{3}\, I_0 \sin(\theta - \phi + 30), \quad . \quad . \quad . \quad . \quad (237)$$

and its effective value is

$$I_A = \sqrt{3} \frac{I_0}{\sqrt{2}} = \sqrt{3}\, I.$$

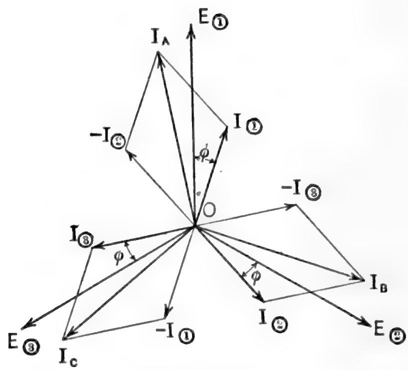


Fig. 204.

The current in line B is

$$i_B = i_2 - i_3 = I_0 \sin(\theta - \phi - 120) - I_0 \sin(\theta - \phi - 240)$$
$$= \sqrt{3}\, I_0 \sin(\theta - \phi - 90), \quad . \quad . \quad . \quad . \quad . \quad (238)$$

and its effective value is

$$I_B = \sqrt{3}\, I.$$

The current in line C is

$$i_c = i_3 - i_1 = I_0 \sin(\theta - \phi - 240) - I_0 \sin(\theta - \phi)$$
$$= \sqrt{3}\, I_0 \sin(\theta - \phi - 210), \quad . \quad . \quad . \quad . \quad (239)$$

and its effective value is

$$I_c = \sqrt{3}\, I.$$

Thus the currents in the three lines are equal in magnitude and

are 120 degrees out of phase with one another. If the effective value of the current in each of the lines is represented by I_l then

$$I_l = I_A = I_B = I_c = \sqrt{3}\, I.$$

The effective values of all these quantities are shown in the vector diagram in Fig. 204.

The power supplied by the alternator is

$$\begin{aligned} P &= E_1 I_1 \cos\phi + E_2 I_2 \cos\phi + E_3 I_3 \cos\phi \\ &= 3\, EI \cos\phi \qquad (240) \\ &= \sqrt{3}\, E_t I_l \cos\phi \qquad (241) \end{aligned}$$

and is equal to $\sqrt{3}$ times the product of the terminal e.m.f., the line current and the power factor.

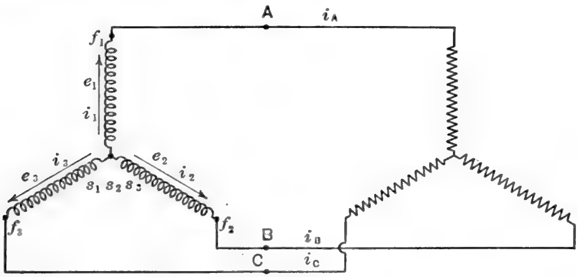


FIG. 205.

If the system is not balanced, that is, if either the currents or the power factors in the three phases differ from one another the line currents will not be equal and they will not be displaced in phase by 120 degrees.

If the current in phase 1 is

$$i_1 = I_{0_1} \sin(\theta - \phi_1)$$

and the current in phase 2 is

$$i_2 = I_{0_2} \sin(\theta - \phi_2 - 120)$$

the current in line A is

$$\begin{aligned} i_A &= i_1 - i_2 \\ &= I_{0_1} \sin(\theta - \Phi_1) - I_{0_2} \sin(\theta - \phi_2 - 120). \qquad (242) \end{aligned}$$

Fig. 205 shows a "Y"-connected three-phase system.

Using the same notation as before and taking the results obtained in Art. 141,

the e.m.f. between lines A and B is

$e_{AB}=e_1-e_2=\sqrt{3}\,E_0 \sin(\theta+30)$, effective value $E_{AB}=\sqrt{3}E=E_l$;

the e.m.f. between lines B and C is

$e_{BC}=e_2-e_3=\sqrt{3}\,E_0 \sin(\theta-90)$, effective value $E_{BC}=\sqrt{3}E=E_l$;

the e.m.f. between lines C and A is

$e_{CA}=e_3-e_1=\sqrt{3}\,E_0 \sin(\theta-210)$, effective value $E_{CA}=\sqrt{3}E=E_l$;

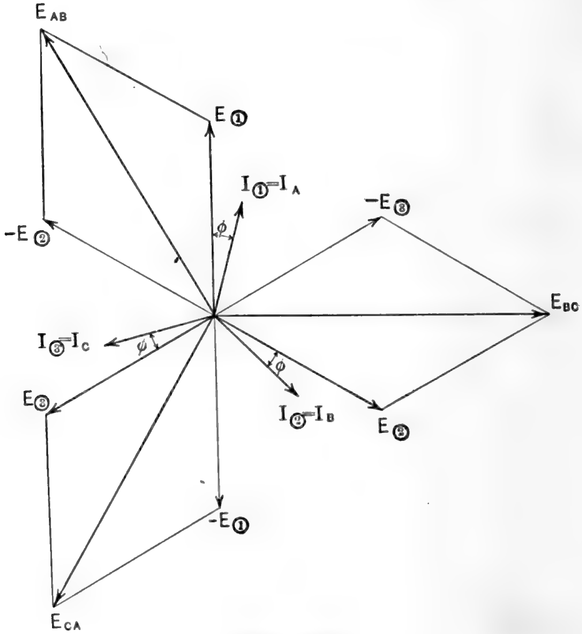


FIG. 206.

the current in line A is

$$i_A = i_1 = I_0 \sin(\theta-\phi), \text{ effective value } I_A = \frac{I_0}{\sqrt{2}} = I_l;$$

the current in line B is

$$i_B = i_2 = I_0 \sin(\theta-\phi-120), \text{ effective value } I_B = \frac{I_0}{\sqrt{2}} = I_l;$$

the current in line C is

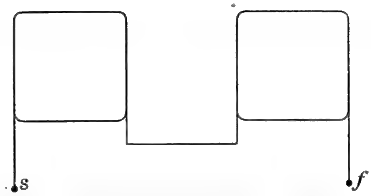
$$i_C = i_3 = I_0 \sin(\theta-\phi-240), \text{ effective value } I_C = \frac{I_0}{\sqrt{2}} = I_l.$$

The effective values of these quantities are all shown in the vector diagram in Fig. 206.

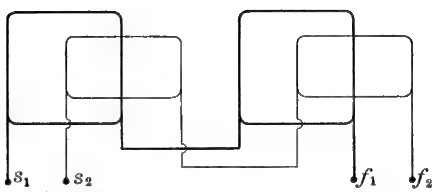
The power supplied by the alternator is as before

$$P = 3\,EI\cos\phi$$
$$= \sqrt{3}\,E_t I_l \cos\phi.$$

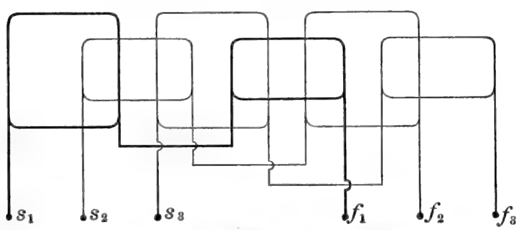
143. Alternator Windings. There are a great many special alternator windings but the majority of them come under the two classes of chain windings and double-layer windings.



Four Pole, Single Phase, Chain Winding



Four Pole, Two Phase, Chain Winding



Four Pole, Three Phase, Chain Winding

FIG. 207. Concentrated chain windings.

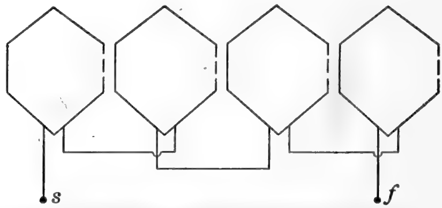
Fig. 207 shows four-pole single-, two- and three-phase chain windings for armatures with one slot per phase per pole. These windings are all concentrated windings.

Fig. 208 shows the corresponding double-layer windings.

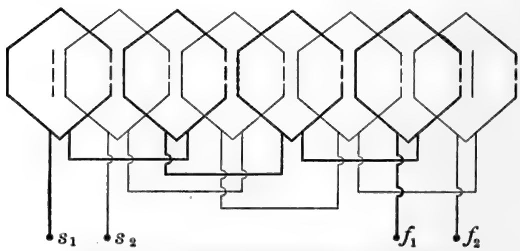
Fig. 209 shows a four-pole single-phase chain winding distributed

in six slots per pole. Fig. 210 shows a winding for the same machine using only four of the six slots per pole as explained in Art. 144.

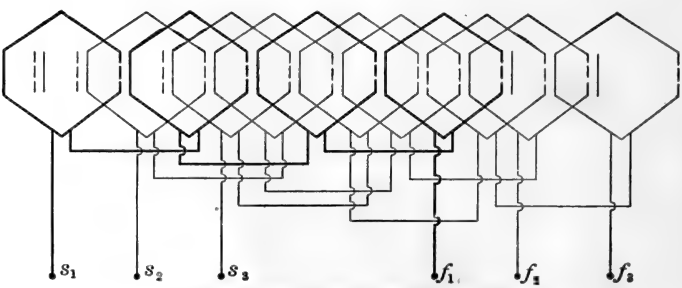
Fig. 211 shows a two-phase chain winding for the armature in Fig. 209. The windings are distributed in three slots per phase per pole.



Four Pole, Single Phase, Double Layer Winding



Four Pole, Two Phase, Double Layer Winding



Four Pole, Three Phase, Double Layer Winding

Fig. 208. Concentrated double-layer windings.

Fig. 212 shows a three-phase chain winding for the same armature, distributed in two slots per phase per pole.

Figs. 213 to 216 show the double-layer windings corresponding to Figs. 209 to 212.

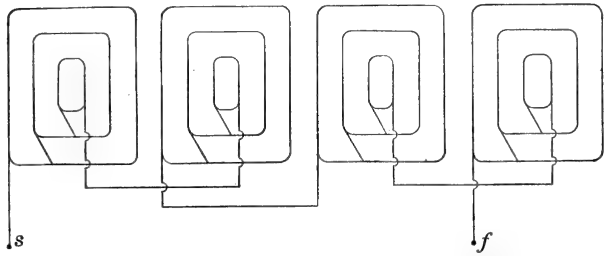


FIG. 209. Four-pole, single-phase, chain winding distributed in six slots per pole.

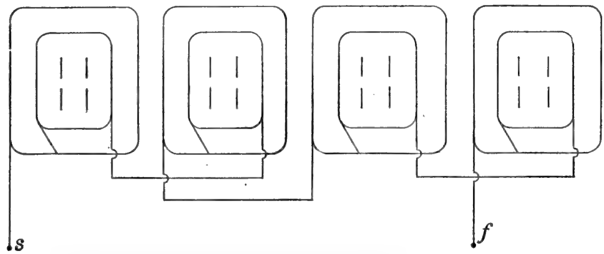


FIG. 210. Four-pole, single-phase, chain winding using only four of the six slots per pole.

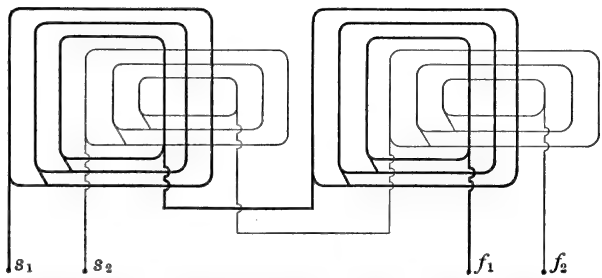


FIG. 211. Four-pole, two-phase chain winding distributed in three slots per phase per pole.

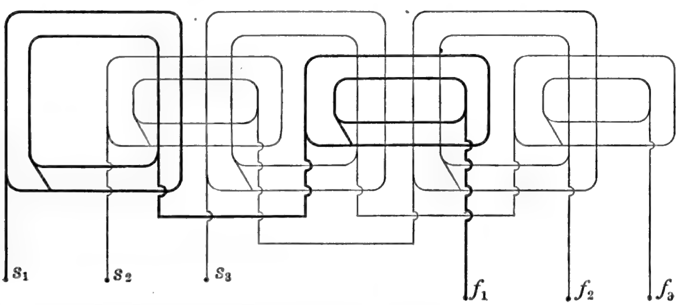


FIG. 212. Four-pole, three-phase, chain winding distributed in two slots per phase per pole.

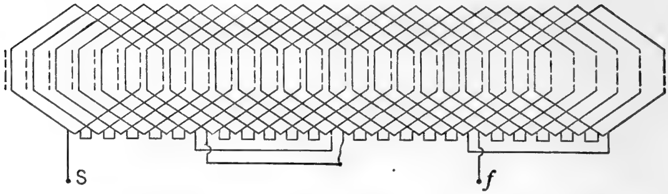


FIG. 213. Four-pole, single-phase, double-layer winding distributed in six slots per pole.

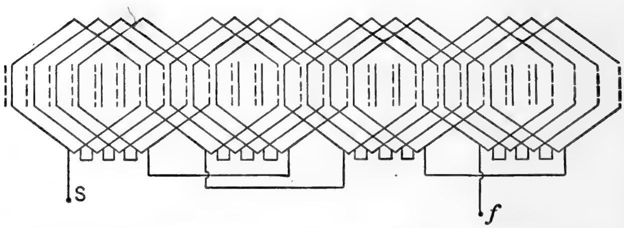


FIG. 214. Four-pole, single-phase, double-layer winding using only four of the six slots per pole.

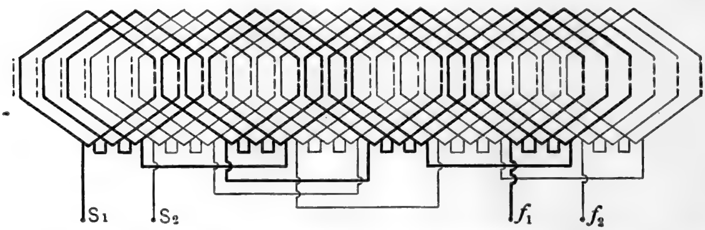


FIG. 215. Four-pole, two-phase, double-layer winding distributed in three slots per phase per pole.

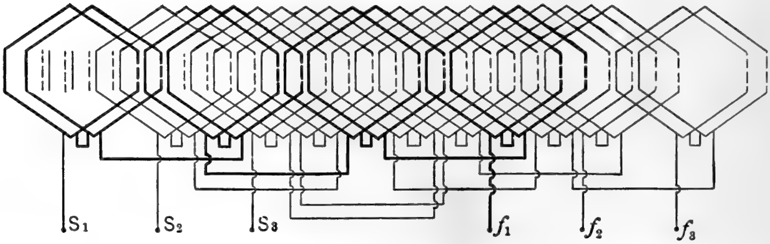


FIG. 216. Four-pole, three-phase, double-layer winding distributed in two slots per phase per pole.

144. Distribution Factors. The windings in Figs. 207 and 208 are all concentrated windings, that is, they are placed in one slot per phase per pole.

When a winding is made up of a number of coils placed in separate slots the e.m.f.'s generated in the various coils are displaced in phase and the terminal e.m.f. is less than if the winding had been concentrated. The factor by which the e.m.f. of a concentrated winding must be multiplied to give the e.m.f. of a distributed winding of the same number of turns is called the distribution factor for the winding and it is always less than unity.

When a single-phase winding is distributed in two slots per pole spaced at 90 degrees the e.m.f.'s in the two coils are 90 degrees out of phase. If the effective value of the e.m.f. generated in each coil is e, then the terminal e.m.f. is $e_t = \sqrt{2}\,e$, Fig. 217, and the distribution factor is

$$\delta = \frac{e_t}{2\,e} = \frac{\sqrt{2}\,e}{2\,e} = 0.705.$$

When a single-phase winding is distributed in three slots per pole spaced at 60 degrees the terminal e.m.f. is the sum of three e.m.f.'s

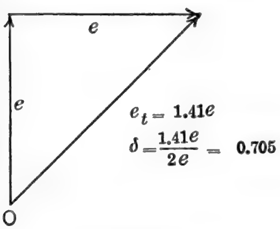


Fig. 217.

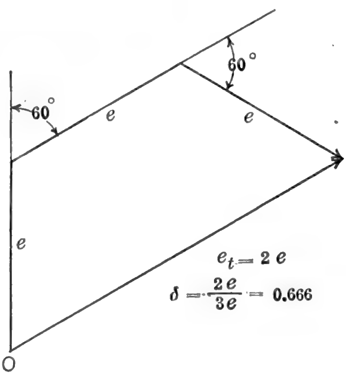


Fig. 218.

e at 60 degrees to one another. It is $e_t = 2\,e$, Fig. 218, and the distribution factor is

$$\delta = \frac{e_t}{3\,e} = \frac{2\,e}{3\,e} = 0.666.$$

When a single-phase winding is distributed in six or more slots per pole the distribution factor may be taken as

$$\delta = \frac{2}{\pi} = 0.637.$$

In Fig. 219 the semi-circumference represents the e.m.f. of the concentrated winding and the diameter represents the e.m.f. of the distributed winding.

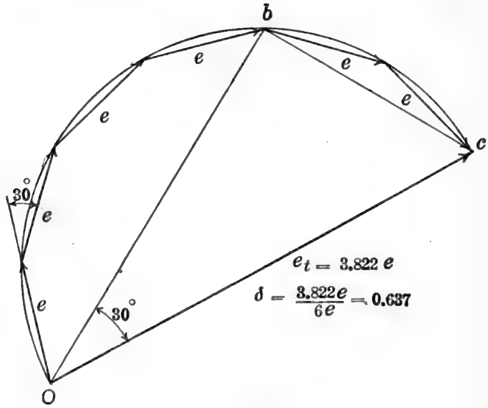


FIG. 219.

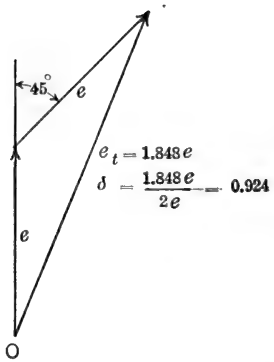


FIG. 220.

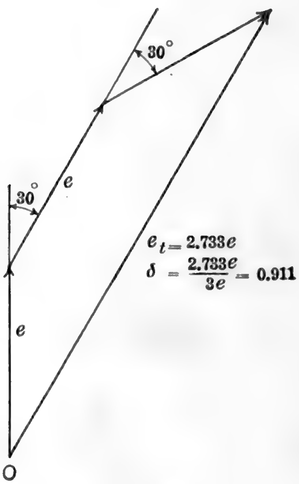


FIG. 221.

The e.m.f.'s in the coils from $b - c$ add very little to the terminal e.m.f. and this part of the winding is usually omitted and the terminal e.m.f. is decreased in the ratio $\frac{ob}{oc} = \cos 30° = 0.866$, or is decreased 13.4 per cent while the resistance and reactance of the winding are decreased $33\frac{1}{3}$ per cent.

Figs. 210 and 214 show four-pole single-phase windings with only four of the six slots per pole used.

The terminal e.m.f. of a two-phase winding distributed in two slots per phase per pole is made up of two e.m.f.'s of value e displaced 45 degrees from one another. It is $e_t = 1.848\ e$, Fig. 220, and the distribution factor is

$$\delta = \frac{e_t}{2\,e} = \frac{1.848\,e}{2\,e} = 0.924.$$

For a two-phase winding with three slots per phase per pole, Figs. 211 and 215, the terminal e.m.f. is made up of three e.m.f.'s displaced 30 degrees from one another. It is $e_t = 2.733\ e$, Fig. 221, and the distribution factor is

$$\delta = \frac{e_t}{3\,e} = \frac{2.733\,e}{3\,e} = 0.911.$$

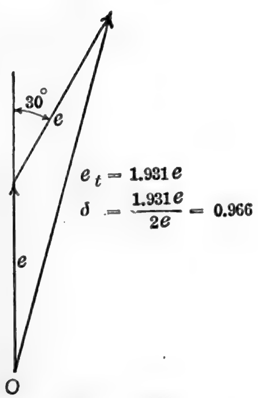


Fig. 222.

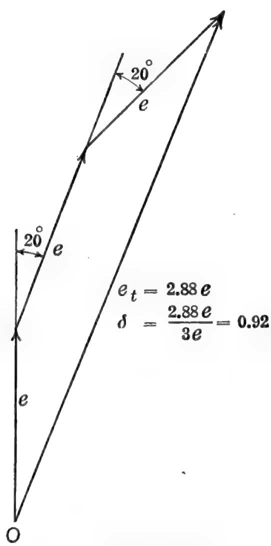


Fig. 223.

The terminal e.m.f. of a three-phase winding distributed in two slots per phase per pole, Figs. 212 and 216, is made up of two e.m.f.'s of value e displaced 30 degrees from one another. It is $e_t = 1.931\ e$, Fig. 222, and the distribution factor is

$$\delta = \frac{e_t}{2\,e} = \frac{1.931\,e}{2\,e} = 0.966.$$

With three slots per phase per pole the factor is

$$\delta = 0.96, \text{ Fig. } 223.$$

The following table gives the distribution factors for single-, two- and three-phase windings.

Slots per phase per pole	Distribution factor		
	Single-phase	Two-phase	Three-phase
1	1.0	1.0	1.0
2	0.705	0.924	0.966
3	0.666	0.911	0.96
4	0.653	0.906	0.958
6	0.637	0.903	0.956

If only two thirds of the slots are used for the single-phase windings their distribution factors must be reduced by multiplying them by the constant 0.866.

145. Short-pitch Windings. The pitch of a winding is the distance between the two sides of one of the coils forming the

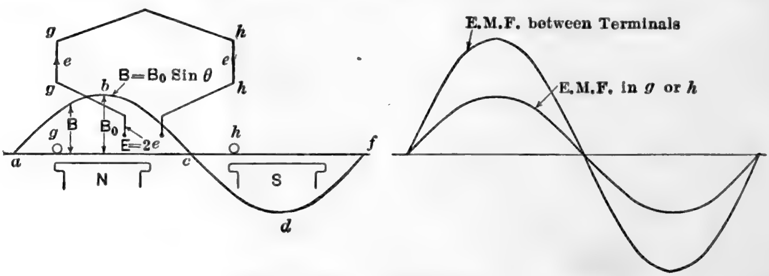


FIG. 224. Full-pitch coil.

winding. When the coil pitch is equal to the pole pitch or the distance between the centres of adjacent poles, the winding is full pitch. When the coil pitch is less than the pole pitch the winding is fractional pitch or short pitch.

In Fig. 224 *abcdf* shows the distribution of flux under two adjacent poles of an alternator. The area under the section of the curve *abc* or *cdf* multiplied by the length of the pole parallel to the shaft gives the flux Φ crossing the air gap under each pole.

As the side *g* of the coil cuts across the flux in the gap an e.m.f. is generated in it of the same wave shape as the flux distribution.

If gh is a full-pitch coil the side h will occupy a position under the adjacent pole similar to that of g and the e.m.f.'s generated in the two sides will be of the same value and wave shape but displaced 180 degrees in phase; they therefore act in the same direction around the coil and add directly to give the terminal e.m.f. If e is the effective value of the e.m.f. generated in one side of the coil the terminal e.m.f. is $E = 2\,e$. With a full-pitch concentrated winding the wave form of the generated e.m.f. is the same as the wave of flux distribution under the poles.

If the coil pitch is less than the pole pitch by an angle α, the e.m.f. wave generated in the side h leads the e.m.f. in g by an

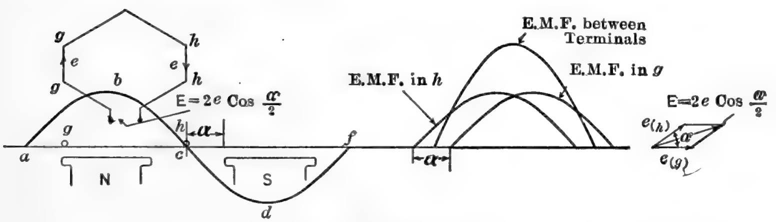


FIG. 225. Short-pitch coil.

angle α, Fig. 225, and the terminal e.m.f. is the vector sum of two e.m.f.'s of effective value e displaced in phase by an angle α. It is

$$E = 2\,e \cos\frac{\alpha}{2}, \qquad \ldots\ldots\ldots \qquad (243)$$

and is less than the e.m.f. generated in the full-pitch winding in the ratio $\cos\frac{\alpha}{2} : 1$.

Fractional-pitch windings are sometimes used in order to eliminate certain harmonics from the e.m.f. wave of the generator. Take the case of a machine with the wave of flux distribution, shown in Fig. 226, consisting of a fundamental and a fifth harmonic. With a full-pitch winding the e.m.f. wave would consist of a fundamental and the prominent fifth harmonic. If, however, the coil pitch is made only 80 per cent of the pole pitch the e.m.f. in one side of the coil will lead that in the other by 36 degrees and the fifth harmonics in the two sides will be in direct opposition and will disappear. (Fig. 227.) The terminal e.m.f. will consist only of the fundamental and it will be decreased in ratio $\cos 18° : 1$. To eliminate an nth harmonic the coil pitch must be

either lengthened or shortened by $\frac{1}{n}$ th of the pole pitch. Thus the wave form of the e.m.f. generated in a short-pitch winding is not the same as the wave of the flux distribution in the air gap.

Similarly the wave form of any distributed winding differs from the wave of flux distribution, since the terminal e.m.f. is the sum of a number of waves displaced from one another. A fully distributed winding gives an e.m.f. wave of approximately sine form at no load regardless of the flux distribution.

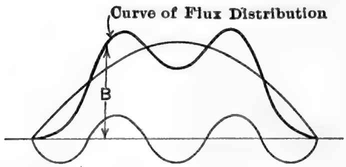


FIG. 226. Flux wave with fifth harmonic.

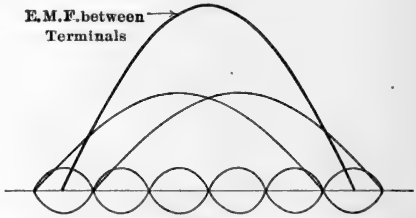


FIG. 227. Elimination of fifth harmonic.

146. Effects of Distributing the Winding. (1) The core is used to better advantage since a number of small slots evenly spaced are used instead of a few large ones. (2) The copper is evenly distributed over the armature surface and thus the copper loss is also distributed and the heat developed by it can more easily be dissipated. A higher current density in the copper can, therefore, be used. (3) The self-inductive reactance is very largely decreased by distributing the winding in a large number of slots, since the coefficient of inductance of a coil is proportional to the square of the number of turns. (4) The terminal e.m.f. is decreased as shown in Art. 144 but the wave form is made more nearly sinusoidal.

147. Multiple-circuit Windings. The windings already discussed are all single circuit, that is, all the turns of one phase are connected in series. In low-voltage machines with a large current output it is necessary to connect the coils forming each phase in multiple circuit. When connected two-circuit the terminal e.m.f. is reduced to one half and the current output is doubled; the power output, therefore, remains the same.

The e.m.f.'s generated in the sections of the windings which are connected in multiple must be of the same value and must be in phase or circulating currents will flow. It is also necessary that

the resistances and reactances of the sections be of the same value or one part of the winding will supply more current than the other.

Fig. 228 shows a four-pole three-phase double-layer winding

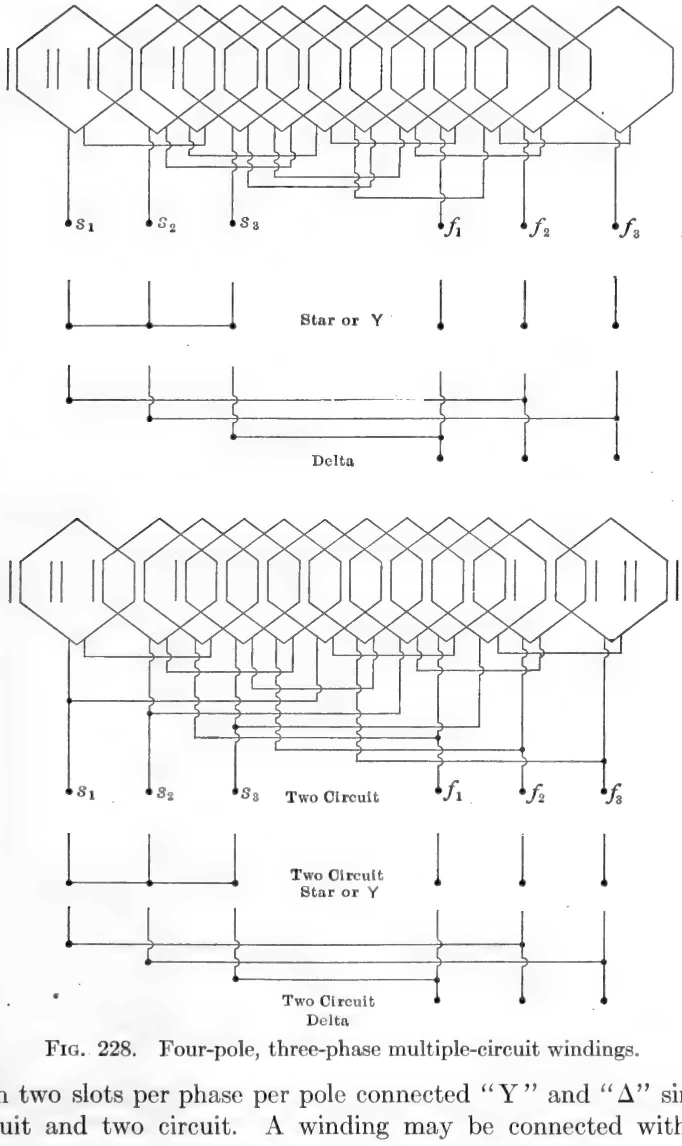


Fig. 228. Four-pole, three-phase multiple-circuit windings.

with two slots per phase per pole connected "Y" and "Δ" single circuit and two circuit. A winding may be connected with as many circuits in multiple as there are pairs of poles.

148. General E.M.F. Equation. The electromotive force equation, derived in Art. 139,

$$E = 4\,\gamma f n \Phi\, 10^{-8} \text{ volts}$$

which applies only to concentrated windings may be extended to include all windings by introducing the distribution factor δ.

Thus the general equation for the effective value of the e.m.f. between terminals of an alternator is

$$E = 4\,\delta\gamma f n \Phi\, 10^{-8} \text{ volts}, \quad . \quad . \quad . \quad . \quad . \quad (244)$$

where

f = frequency in cycles per second,
n = number of turns in series between terminals,
Φ = flux from one pole,
γ = form factor of the e.m.f. wave,
δ = distribution factor of the winding.

This equation holds both for the single-phase alternator and for any phase of a polyphase alternator with n turns in series per phase.

If the winding is short pitch the e.m.f. is reduced in the ratio $\cos\frac{\alpha}{2} : 1$ where the coil pitch is $180 - \alpha$ electrical degrees.

149. Rating of Alternators. Alternators are designed to give a certain terminal voltage and to supply any current up to a certain maximum or full-load current.

The output is

$$P = nEI \cos\theta \text{ watts},$$

where

E is the voltage per phase,
I is the full-load current per phase,
$\cos\theta$ is the power factor of the load, and
n is the number of phases.

The power output, therefore, depends on the voltage which is a fixed quantity, the current which is variable and is limited by the allowable temperature rise caused by the copper losses and other losses in the machine, and the power factor of the load over which the designer has no control.

Alternators should, therefore, be rated not in watts or kilowatts which depend on the power factor but in volt amperes or kilovolt amperes.

A machine rated at 1000-kilovolt amperes can supply 1000 kilo-

watts to a non-inductive load at unity power factor or it can supply $1000 \times 0.80 = 800$ kilowatts to an inductive load of 80 per cent power factor.

150. Comparative Ratings of an Alternator Wound Single-, Two- and Three-Phase. Take the case of a machine with six slots per pole. Let e be the effective value of the e.m.f. generated in each coil of the winding and I be the current per conductor. The current will be the same in the three cases for the same temperature rise.

When wound single-phase using all the slots the distribution factor is 0.64 and the terminal e.m.f. is

$$E = 6\,e \times 0.64$$

and the output is

$$P_1 = EI \cos \phi = 3.84\,eI \cos \phi,$$

where $\cos \phi$ is the power factor of the load.

When wound single-phase using only four slots per pole the terminal e.m.f. is

$$E = 6\,e \times 0.64 \times 0.866 \text{ (Art. 144)}$$

and the output is

$$P_1' = EI \cos \phi = 3.32\,eI \cos \phi.$$

When wound two-phase with three slots per phase per pole the distribution factor is 0.91, the e.m.f. per phase is

$$E = 3\,e \times 0.91$$

and the output is

$$P_2 = 2\,EI \cos \phi = 5.46\,eI \cos \phi.$$

When wound three-phase with two slots per phase per pole the distribution factor is 0.96, the e.m.f. per phase is

$$E = 2\,e \times 0.96$$

and the output is

$$P_3 = 3\,EI \cos \phi = 5.76\,eI \cos \phi.$$

Taking the three-phase rating as 100 the comparative ratings are as given below.

Number of Phases.	Rating.
Three-phase	100
Two-phase	95
Single-phase using all the slots	67
Single-phase using only four slots per pole	57.7

In practice an alternator is given the same rating two- and three-phase and 65 per cent of that rating single-phase.

151. Armature Reaction. The flux distribution in the air gap of an alternator at no load is symmetrical about the centre line of the pole and usually follows approximately a sine wave. The e.m.f. generated in the armature is also a sine wave. (See Fig. 224.)

When current flows in the armature winding, the m.m.f. of the armature combines with the m.m.f. of the field and changes both the magnitude and distribution of the flux crossing the air gap and cut by the armature conductors. It thus changes both the magnitude and the wave form of the e.m.f. generated. These results are termed the armature reaction.

Armature reaction depends not only on the intensity of the current in the armature but also on its phase relation with the generated e.m.f. Fig. 229 illustrates armature reaction in a machine with a single-phase concentrated winding.

In (*a*) the armature coil is shown in the position of zero e.m.f.; if the current is in phase it is also zero.

In (*b*) the e.m.f. is maximum and the current is maximum. The m.m.f. of the armature is cross magnetizing, that is, it decreases the flux over one half of the pole and increases it over the other half. The useful flux is only decreased by the small amount lost due to the higher saturation, and, therefore, decreased permeability over the half of the pole where the density is increased. The flux distribution no longer follows a sine wave and the e.m.f. will not be a sine wave.

In (*c*) the current is maximum but lags 90 degrees behind the generated e.m.f. The m.m.f. of the armature acts directly against the m.m.f. of the field. It is, therefore, demagnetizing and decreases the flux but does not distort it.

In (*d*) the current is maximum and leads the e.m.f. by 90 degrees. The m.m.f. of the armature acts directly with the field and magnetizes it. The useful flux is increased and is not distorted.

The following results have been obtained:

(1) A current in phase with the generated e.m.f. is cross magnetizing and only decreases the flux to a very slight extent.

(2) A current lagging 90 degrees behind the generated e.m.f. demagnetizes the field and decreases the flux and decreases the generated e.m.f.

(3) A current leading the generated e.m.f. by 90 degrees magnetizes the field, increases the flux and increases the generated e.m.f.

If the current lags behind the e.m.f. by angle ϕ, it may be resolved into two components, $I \cos \phi$ in phase with the e.m.f.

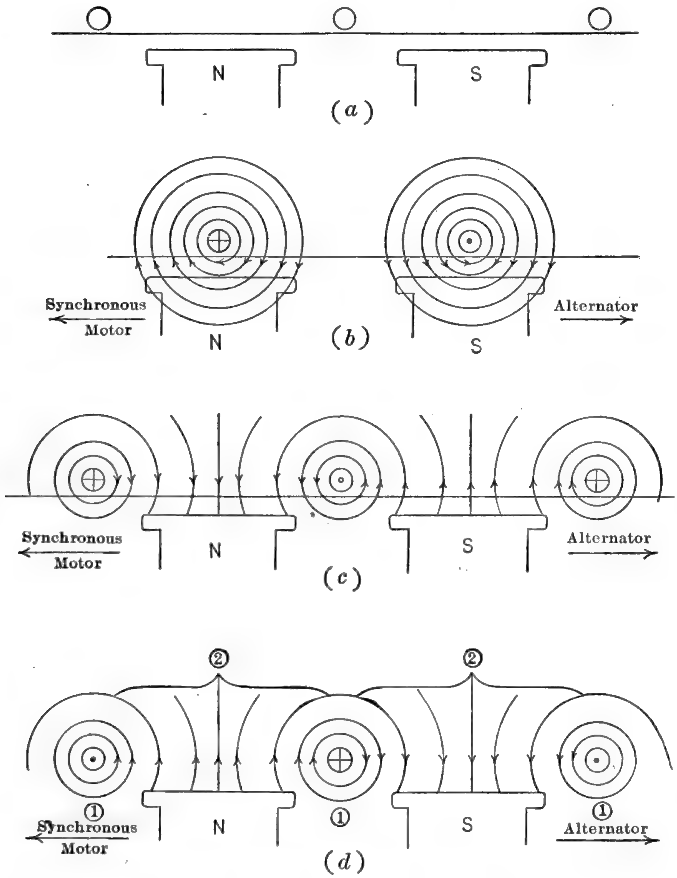


FIG. 229. Armature reaction.

and, therefore, cross magnetizing and $I \sin \phi$ lagging 90 degrees behind the e.m.f. and demagnetizing.

The effect of armature reaction increases almost directly with the current. When the magnetic circuit, or that part of it including the pole tips, air gaps and teeth, becomes saturated the

cross-magnetizing effect is decreased since a large m.m.f. is required to produce a small change in flux. The demagnetizing m.m.f. increases directly with the current and the leakage factor of the machine also increases and causes a further decrease of the useful flux.

152. Armature Reactance. The flux produced by the current in the armature coil in Fig. 229 may be separated into two parts as shown.

Part (1) is the flux of armature reaction which crosses the gap and interferes with the flux threading the field circuit. Its effect is either cross magnetizing, demagnetizing or magnetizing.

Part (2) is the flux which only interlinks with the coil itself and does not interfere with the flux produced by the field m.m.f. It is the self-inductive flux of the coil and generates in the coil an e.m.f. of self-inductance, which consumes a component of the e.m.f. generated by rotation. This e.m.f. called the armature reactance drop is equal to the product of the armature current I and the armature reactance x and leads the current by 90 degrees.

The reactance is $x = 2\pi fL$, where L is the inductance of the armature. L and x both decrease as the armature current increases due to the increased saturation and, therefore, decreased permeability of the leakage path surrounding the armature conductors. They also vary as the armature is rotated; when the conductor is under the pole the reluctance of its local leakage path is minimum and L and x are large; when between the poles the reluctance is maximum and L and x are reduced. An average value of x is chosen to represent the armature reactance. At light loads when the current is small the reactance drop will be greater than the value corresponding to the average reactance and when the current is large it will be smaller.

153. Polyphase Armature Reaction. If n is the number of turns per phase per pair of poles on a two-phase alternator and $i_1 = I_0 \cos \theta$ is the current in phase 1 and $i_2 = I_0 \cos (\theta - 90°) = I_0 \sin \theta$ is the current in phase 2, the m.m.f.'s of the two phases are $m_1 = ni_1 = nI_0 \cos \theta$ and $m_2 = ni_2 = nI_0 \sin \theta$. These two m.m.f.'s are in quadrature in time and space but combine to give a constant m.m.f. nI_0 fixed in position relative to the field m.m.f. and revolving synchronously backwards relative to the armature. This can be seen by reference to Fig. 230. AB is the winding of phase 1 and the current is assumed to lag behind the e.m.f. by

angle ϕ. At the instant represented the current is maximum and the m.m.f. of the coil is maximum and acts in direction OY. The current in phase 2 is now zero. The m.m.f.'s acting are shown in Fig. 231.

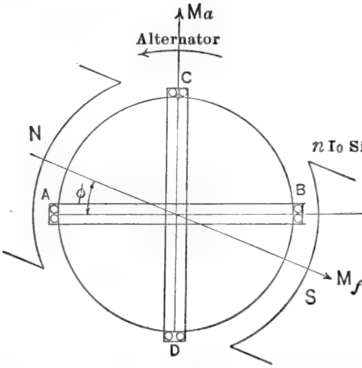


FIG. 230.

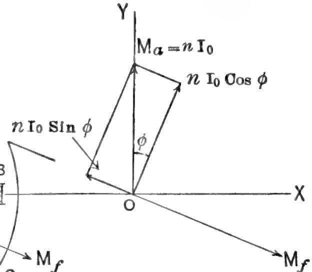


FIG. 231.

In Fig. 232 the coil AB has moved through angle θ and its current has decreased to $I_0 \cos \theta$ and its m.m.f. to $nI_0 \cos \theta$. The current in coil CD has a value $I_0 \sin \theta$ and its m.m.f. is $nI_0 \sin \theta$.

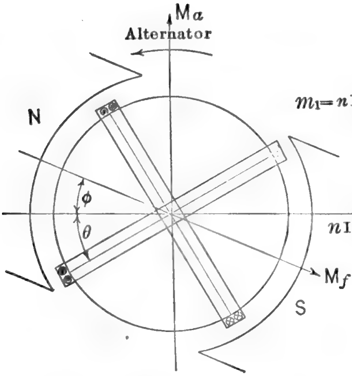


FIG. 232.

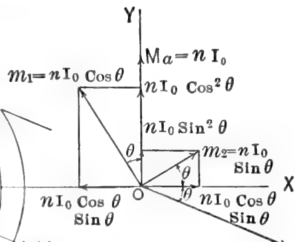


FIG. 233.

The component of the m.m.f. of phase 1 in direction OY is $nI_0 \cos \theta \cdot \cos \theta = nI_0 \cos^2 \theta$ and the component in direction OX is $-nI_0 \cos \theta \cdot \sin \theta$.

The component of the m.m.f. of phase 2 in direction OY is $nI_0 \sin \theta \cdot \sin \theta = nI_0 \sin^2 \theta$ and the component in direction OX is $+nI_0 \sin \theta \cdot \cos \theta$. The resultant m.m.f. of the two phases in direc-

tion OY is $nI_0 \cos^2 \theta + nI_0 \sin^2 \theta = nI_0$ and in the direction OX is $nI_0 \cos \theta \cdot \sin \theta - nI_0 \cos \theta \cdot \sin \theta = 0$.

Thus the resultant armature m.m.f. is nI_0 in fixed direction relative to the field m.m.f. and, therefore, revolving synchronously relative to the armature. (Fig. 233.)

The direction of the resultant armature m.m.f. relative to the field m.m.f. is determined by the angle of phase difference between the current and the e.m.f. generated at no load.

If the current is in phase with the e.m.f., the armature m.m.f. acts at right angles to the field m.m.f. and is, therefore, cross magnetizing only; if the current lags by angle ϕ, the armature m.m.f. can be separated into two components $nI_0 \sin \phi$ which is demagnetizing and $nI_0 \cos \phi$ which is cross magnetizing. (Fig. 231.)

If the two-phase winding, Fig. 232, is replaced by a three-phase winding, Fig. 234, with the first phase AB in the same position as

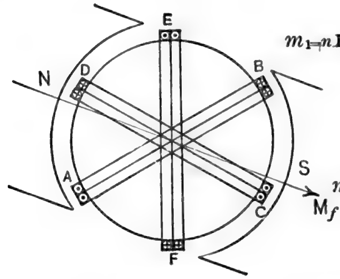


FIG. 234.

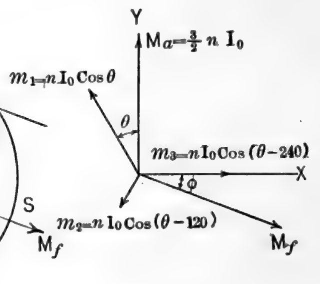


FIG. 235.

before and the other phases CD and EF displaced 120 degrees and 240 degrees from it, the m.m.f.'s of the three phases will be respectively $nI_0 \cos \theta$, $nI_0 \cos (\theta - 120)$ and $nI_0 \cos (\theta - 240)$ and will act in the directions represented. As before θ is measured from the instant of maximum current and the currents in the three phases are assumed to lag behind the e.m.f.'s by angle ϕ.

The sum of the components of m.m.f. in direction OY is

$$nI_0 \cos^2 \theta + nI_0 \cos^2 (\theta - 120) + nI_0 \cos^2 (\theta - 240) = \tfrac{3}{2}\, nI_0,$$

and the sum of the components in the direction OX is

$$nI_0 \cos \theta \sin \theta + nI_0 \cos (\theta - 120) \sin (\theta - 120) \\ + nI_0 \cos (\theta - 240) \sin (\theta - 240) = 0.$$

Thus, the resultant m.m.f. of the armature of a three-phase alternator is

$$M_a = \tfrac{3}{2}\, nI_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (246)$$

where n is the number of turns in series per phase and I_0 is the maximum value of the armature current.

The armature m.m.f. is fixed in direction relative to the fields and revolves synchronously relative to the armature. (Fig. 235.)

154. Single-phase Armature Reaction. If n is the number of turns per pair of poles on the armature of a single-phase alternator and $i = I_0 \sin(\theta - \phi)$ is the instantaneous value of the armature current at time t and angle θ after the position of zero e.m.f., the m.m.f. of the armature is $m_a = nI_0 \sin(\theta - \phi)$. (Fig. 236.) It does not revolve relative to the armature and remain fixed relative to the field as in the polyphase machine but revolves with the armature and pulsates between zero and a maximum value nI_0, producing a double-frequency pulsation of the field and a third harmonic of e.m.f. in the armature.

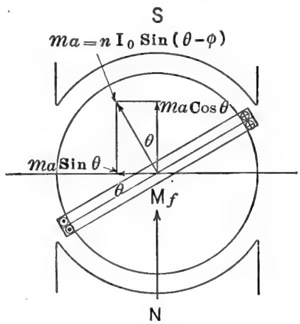


FIG. 236.

The armature m.m.f. may be resolved into two components, one at right angles to the field m.m.f. and the other in line with it.

The first component is $m_a \sin\theta = nI_0 \sin(\theta - \phi) \sin\theta$ and is cross magnetizing only. It does not act directly on the magnetic circuit and only decreases the useful flux due to increased saturation. Its effect will be neglected in the following discussion:

The second component $m_a \cos\theta = nI_0 \sin(\theta - \phi)\cos\theta$ is directly in line with the field m.m.f. and produces a double-frequency pulsation of the flux linking the magnetic circuit.

The total m.m.f. acting on the magnetic circuit at time t and angle θ is

$$\begin{aligned} m &= M_f + m_a \cos\theta \\ &= M_f + nI_0 \sin(\theta - \phi)\cos\theta \\ &= M_f + \frac{nI_0}{2}\{\sin(2\,\theta - \phi) - \sin\theta\} \\ &= M_f - \frac{nI_0}{2}\sin\phi + \frac{nI_0}{2}\sin(2\,\theta - \phi). \quad . \quad . \quad . \qquad (247) \end{aligned}$$

The second term represents a constant demagnetizing effect and the last term is a double-frequency quantity.

If the air-gap length is so adjusted that its permeance varies according to a sine law from a maximum $\mathfrak{P}_0$ under the centre of a pole to zero, midway between poles, Fig. 237, its equation is

$$\mathfrak{P} = \mathfrak{P}_0 \sin \theta. \qquad (248)$$

The flux density in the gap at any angle θ is

$$\mathfrak{B} = m\mathfrak{P} = \mathfrak{P}_0 \left\{ \left(M_f - \frac{nI_0}{2} \sin \phi \right) \sin \theta + \frac{nI_0}{2} \sin (2\,\theta - \phi) \sin \theta \right\}$$

lines per sq. cm. $\qquad (249)$

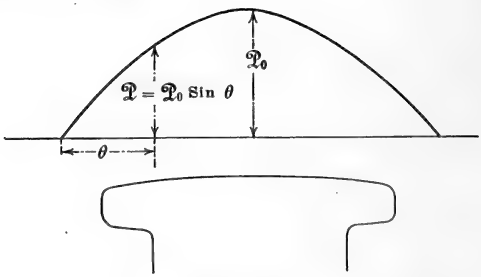


Fig. 237.

The flux cut per second by the two sides of the coil is $2\,\mathfrak{B}lv$ lines, where l is the length of conductor under the pole in cm. and v is the velocity in cm. per sec.

Thus the e.m.f. generated in the coil of n turns is

$$e = 2\,\mathfrak{B}lvn\, 10^{-8} = K\mathfrak{B} \text{ volts,}$$

where $K = 2\, lvn 10^{-8}$ is a constant,
or substituting for $\mathfrak{B}$

$$\begin{aligned}
e &= K\mathfrak{P}_0 \left\{ \left(M_f - \frac{nI_0}{2} \sin \phi \right) \sin \theta + \frac{nI_0}{2} \sin (2\,\theta - \phi) \sin \theta \right\} \\
&= K\mathfrak{P}_0 \left[\left(M_f - \frac{nI_0}{2} \sin \phi \right) \sin \theta + \frac{nI_0}{4} \{ \cos (\theta - \phi) - \cos (3\,\theta - \phi) \} \right] \\
&= K\mathfrak{P}_0 \left[\left(M_f - \frac{nI_0}{2} \sin \phi \right) \sin \theta + \frac{nI_0}{4} \{ \sin \theta \sin \theta \right. \\
&\qquad \left. + \cos \theta \cos \phi - \cos (3\,\theta - \phi) \} \right] \\
&= K\mathfrak{P}_0 \left\{ \left(M_f - \frac{nI_0}{4} \sin \phi \right) \sin \theta + \frac{nI_0}{4} \cos \phi \cos \theta - \frac{nI_0}{4} \cos \left(3\,\theta - \phi \right) \right\} \\
&= \{ A \sin \theta + B \cos \theta - C \cos (3\,\theta - \phi) \} \qquad (250)
\end{aligned}$$

where A, B and C are constants.

This may be further simplified by combining the first two terms and becomes

$$e = \sqrt{A^2 + B^2} \sin (\theta + \gamma) - C \cos (3\,\theta - \phi), \quad . \quad (251)$$

where

$$\gamma = \cos^{-1} \frac{A}{\sqrt{A^2 + B^2}}.$$

Thus the generated e.m.f. consists of a sine wave of fundamental frequency and a third harmonic.

When $\phi = 90$ degrees, as on short circuit, the third harmonic is

$$\begin{aligned} e_3 &= - C \cos (3\,\theta - 90) \\ &= - C \sin 3\,\theta; \end{aligned}$$

when $\theta = 90$,

$$e_3 = - C \sin 270° = C;$$

therefore, at short circuit, the generated e.m.f. contains a third harmonic, which passes through zero at the same instant as the fundamental wave, and the wave of e.m.f. is symmetrical, with a peak at the centre.

Thus single-phase armature reaction produces a double-frequency pulsation of the field and a third harmonic of e.m.f.

Since the magnetic circuit is surrounded by a field winding, consisting of a large number of turns, the pulsations will be less than the above results indicate. The variation of the flux linking with the field winding induces in it e.m.fs. and currents which oppose the variation and limit it to a small value.

In a machine with a large number of field turns the pulsation produced by armature reaction up to full-load current is very small and the armature reaction may be considered as constant in value with reference to the fields and demagnetizing.

In the case of a short circuit, however, where from three to five times full-load current flows in the armature, a large pulsation of flux is produced in the magnetic circuit and very large e.m.fs. and currents may be induced in the field windings.

If a short circuit occurs on one phase only of a three-phase alternator, the armature reaction can be separated into the ordinary three-phase armature reaction with equal currents and a single-phase armature reaction due to the excess of the short-circuit current over normal current acting in the turns of one phase. This single-phase armature reaction produces a double-frequency pulsation of the field and a third harmonic of e.m.f. in all the phases.

The effects of single-phase armature reaction are relatively greater in machines with a small number of turns on the fields, as turbo alternators.

155. Electromotive Forces in the Alternator. In studying the performance of an alternator it is necessary to determine the relation between the terminal e.m.f. E, the e.m.f. E_1 generated by rotation and the e.m.f. E_0 generated at no load.

E_1 is the e.m.f. generated in the armature by the rotation of the flux produced in the air gap by the resultant of the magnetomotive forces of the field and armature. It is the vector sum of the terminal e.m.f. E and the e.m.f. consumed by the impedance of the armature. The armature impedance is $z = \sqrt{r^2 + x^2}$, or expressed in rectangular coördinates $\underset{.}{z} = r + jx$, where r is the resistance of the armature and consumes a component of e.m.f. Ir in phase with the current I, and x is the true self-inductive reactance of the armature and consumes a component of e.m.f. Ix in quadrature ahead of the current.

The generated e.m.f. thus is

$$\begin{aligned} \underset{.}{E}_1 &= \underset{.}{E} + \underset{.}{I}\underset{.}{z} \\ &= \underset{.}{E} + \underset{.}{I}\,(r + jx), \qquad \qquad (252) \end{aligned}$$

and the terminal e.m.f. is the vector difference between the e.m.f. generated in the armature by rotation and the impedance drop

$$\underset{.}{E} = \underset{.}{E}_1 - \underset{.}{I}\,(r + jx). \qquad \qquad (253)$$

E_0 is the e.m.f. generated at no load due to cutting the flux produced by the field m.m.f. M_f acting alone. Under load current flows in the armature and exerts a m.m.f. M_a, which is either cross magnetizing, demagnetizing or magnetizing depending on the phase relation of the current and the terminal e.m.f. This armature m.m.f. combines with the field m.m.f. and changes both the intensity and the distribution of the flux in the gap, so that under load the e.m.f. generated in the armature is not the same as at no load.

The difference between the two is the e.m.f. consumed or the e.m.f. not generated due to the presence of the armature reactance. This e.m.f. is proportional to the current and can be expressed as the product of the current I and a component of reactance x'. It is Ix' and is in quadrature ahead of the current. Thus

$$\begin{aligned} \underset{.}{E}_0 &= \underset{.}{E}_1 + j\underset{.}{I}x' \\ &= \underset{.}{E} + \underset{.}{I}\,(r + jx) + j\underset{.}{I}x' \\ &= \underset{.}{E} + \underset{.}{I}\,\{r + j\,(x + x')\} \\ &= \underset{.}{E} + \underset{.}{I}\,(r + jx_0). \qquad \qquad (254) \end{aligned}$$

The total reactance of the armature x_0 is called the synchronous reactance and consists of two components, x which represents the effect of the armature leakage flux and which has been called the armature reactance and x' which represents the effect of armature reaction.

The quantity $z_0 = r + jx_0$ is the synchronous impedance of the armature and consumes a component of the no-load e.m.f. $Iz_0 = I\,(r + jx_0)$ which is the synchronous impedance drop in the armature.

Thus the e.m.f. generated at no load is the vector sum of the terminal e.m.f. and the synchronous impedance drop

$$E_0 = E + I\,(r + jx_0).$$

156. Vector Diagram of E.M.F.'s and M.M.F.'s. In Fig. 238 is drawn the vector diagram of an alternator supplying an inductive load.

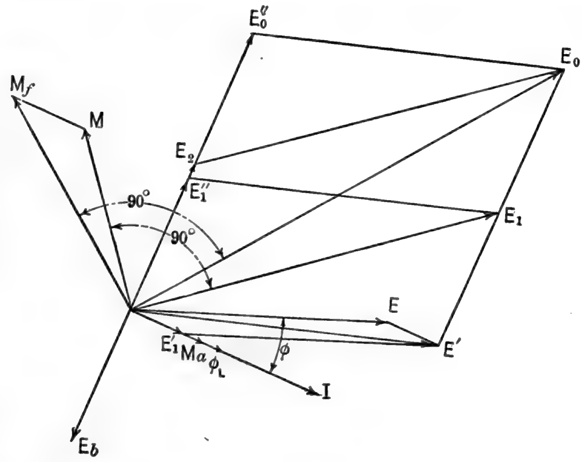


FIG. 238.

$OE \;\;= E \;\;=$ terminal e.m.f.

$OI \;\;= I \;\;=$ armature current lagging behind E by angle ϕ.

$OE_1' = E_1' = Ir =$ e.m.f. consumed by the armature resistance r in phase with I.

$OE' \;= E' \;= E + Ir =$ resultant of all the e.m.f.'s generated in the armature.

$OE_b \;= E_b \;= -jIx =$ e.m.f. generated in the armature by the leakage flux ϕ_L, produced by the armature current.

$OE_1'' = E_1'' = -E_b = jIx$ = e.m.f. consumed by the armature reactance x in quadrature ahead of I.

$OE_1 = E_1 = E + I\ (r + jx)$ = e.m.f. generated by rotation, that is, the e.m.f. generated by cutting the flux produced by the resultant of the m.m.f.'s of field and armature.

$OM_a = M_a$ = m.m.f. of the armature current.

$OM = M$ = resultant of field m.m.f. and armature m.m.f. in quadrature ahead of the e.m.f. E_1 produced by it.

$OM_f = M_f$ = field m.m.f.

$OE_0 = E_0$ = e.m.f. generated in the armature at no load, in quadrature behind the field m.m.f.

The no-load e.m.f. can be separated into two components OE_1 and OE_2.

$OE_2 = E_2 = jIx'$ = e.m.f. consumed by armature reaction, in quadrature ahead of I.

If the two reactance drops Ix and Ix' are combined the diagram may be simplified as shown in Fig. 239.

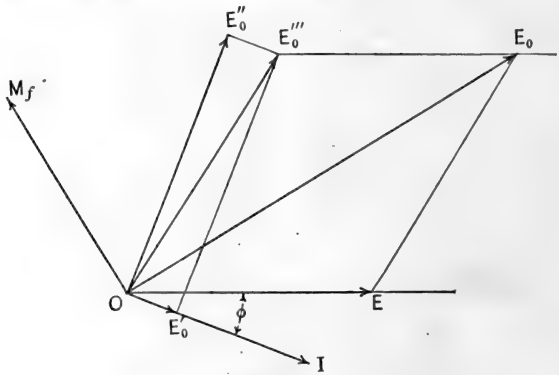


FIG. 239.

$OE = E$ = terminal e.m.f.

$OI = I$ = armature current lagging by angle ϕ.

$OE_0' = E_0' = Ir$ = armature resistance drop in phase with I.

$OE_0'' = E_0'' = jI\ (x + x') = jIx_0$ = synchronous reactance drop in quadrature ahead of I.

$OE_0''' = E_0''' = Iz_0$ = synchronous impedance drop.

$OE_0 = E_0 = E_0 + Iz_0$ = e.m.f. generated at no load.

$OM_f = M_f$ = field m.m.f. in quadrature ahead of E_0 and proportional to it, neglecting the effect of saturation.

In Figs. 240, 241 and 242 are drawn the diagrams for the three cases (1) $\phi = 0$, or non-inductive load, (2) $\phi = 60°$ lag, or inductive load with a power factor of cos 60° = 50 per cent, and (3) $\phi = 60°$ lead, or capacity load with a power factor of 50 per cent.

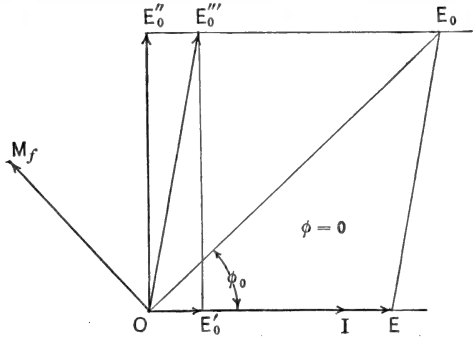


FIG. 240.

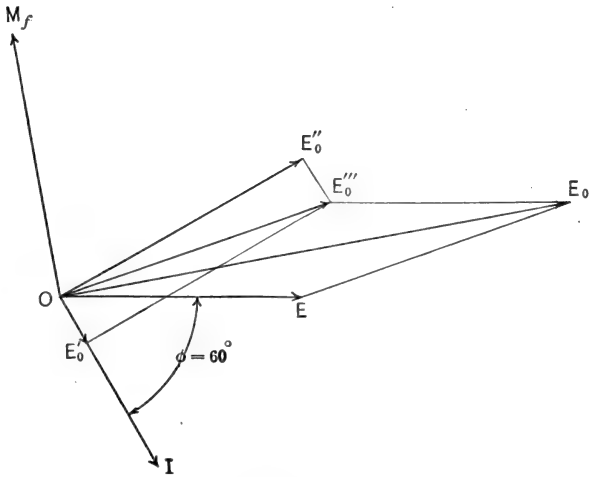


FIG. 241.

The relation between E_0, E and I can be expressed in the form of an equation by reference to Fig. 243, which is a reproduction of Fig. 239.

$$E_0^2 = \overline{ok}^2 + \overline{fk}^2$$
$$= \overline{og + gk}^2 + \overline{fl + lk}^2$$
$$= (E \cos \phi + Ir)^2 + (E \sin \phi + Ix_0)^2$$

or $$E_0 = \sqrt{(E \cos \phi + Ir)^2 + (E \sin \phi + Ix_0)^2}. \quad . \quad (255)$$

The same result can be obtained by using rectangular co-ordinates.

If I is taken as real axis, then the terminal e.m.f. E leading it by angle ϕ is

$$\dot{E} = E \cos \phi + jE \sin \phi.$$

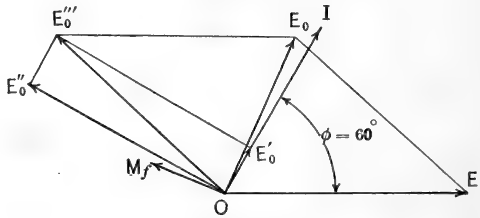


FIG. 242.

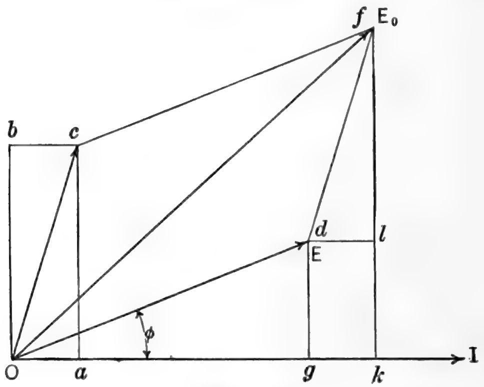


FIG. 243.

The synchronous impedance drop is

$$\dot{E}_0''' = I\,(r + jx_0)$$

and the no-load e.m.f. is

$$\dot{E}_0 = \dot{E} + \dot{E}_0''' = (E \cos \phi + Ir) + j\,(E \sin \phi + Ix_0),$$

or its absolute value is

$$E_0 = \sqrt{(E \cos \phi + Ir)^2 + (E \sin \phi + Ix_0)^2}.$$

By reference to these figures it is seen that for the same terminal e.m.f. E, and the same current I, a much larger value of E_0 is required for inductive loads than for non-inductive loads and a much smaller value for capacity loads than for non-inductive loads.

While the magnetic circuit of the machine is unsaturated the field excitation required to produce the e.m.f. E_0 is approximately proportional to it, but higher up on the saturation curve the excitation increases faster than the e.m.f. The relation between field-exciting current and no-load e.m.f. can be obtained by reference to Fig. 244 which is the saturation curve of the alternator at no load.

157. Determination of the Synchronous Impedance. In order to predetermine the characteristic curves of an alternator it is necessary to know the relation between the field-exciting current and the terminal e.m.f. at no load, given by the saturation curve in Fig. 244, and also to know the components of the synchronous

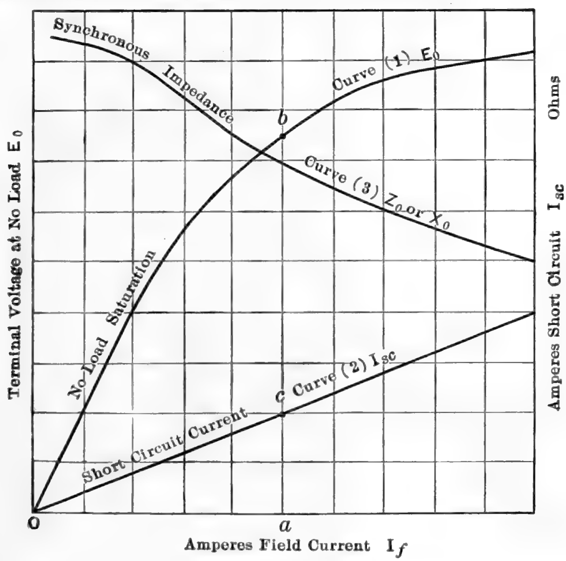


FIG. 244.

impedance of the armature. To obtain these a short-circuit test is made on the alternator as follows: Short circuit the armature through an ammeter and run the alternator at rated frequency but with very low excitation. Gradually increase the excitation until about twice full-load current flows in the armature. Plot the values of short-circuit current I_{sc} on a base of field current I_f, curve (2), Fig. 244. The locus will be a straight line passing through the origin and can be produced beyond the range obtained in the test.

At any value of field current oa, the no-load voltage is ab and the short-circuit current is ac. Since the terminal e.m.f. is zero the e.m.f. ab must represent the synchronous impedance drop due to the current ac. The synchronous impedance is, therefore,

$$z_0 = \frac{E_0}{I_{sc}} = \frac{ab}{ac}.$$

The ordinates of curve (3), Fig. 244, are obtained as the quotients of the corresponding ordinates of curve (1) and curve (2). The

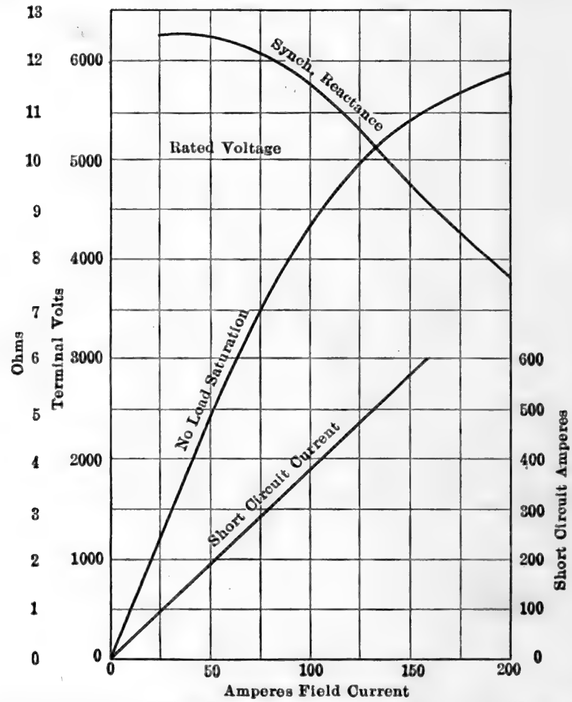


FIG. 245. Characteristic curves of a 2500-kv.a., 5000-volt, three-phase alternator.

ordinates of curve (3), therefore, represent the synchronous impedance of the armature. It is not constant but is high when the magnetic circuit of the alternator is unsaturated and decreases as the excitation is increased and the magnetic circuit becomes saturated.

If the power required to operate the machine is measured and the excitation loss, iron losses and friction losses are subtracted,

the remainder P_0 is the loss in the armature due to the current I_{sc}. The effective resistance of the armature is

$$r = \frac{P_0}{I_{sc}^2}. \qquad (256)$$

It is greater than the true ohmic resistance as measured with direct current since some extra losses called load losses occur in the copper due to eddy currents set up by the unequal distribution of flux throughout the volume of the conductor.

The values of the synchronous reactance can be obtained by separating the resistance from the synchronous impedance according to the equation

$$x_0 = \sqrt{z_0^2 - r^2}. \qquad (257)$$

The resistance r is, however, very small in comparison to the synchronous reactance x_0 and the ordinates of curve (3), Fig. 244, may be taken to represent either x_0 or z_0. For purposes of calculation an average value of x_0 is taken and it is assumed to remain constant.

Fig. 245 shows the results of a test on a 2500-kv.a., 5000-volt, three-phase alternator.

158. Voltage Characteristics. The relation between the terminal e.m.f. and armature current of an alternator, with a fixed value of field current and a given load power factor, is called the "regulation curve" or "voltage characteristic" for the given power factor.

When the power factor is unity and the current is in phase with the terminal e.m.f. E, it lags by an angle ϕ_0 (Fig. 240) behind the no-load e.m.f. E_0 and there is thus a small demagnetizing effect proportional to $I \sin \phi_0$, which decreases the flux and a large cross-magnetizing effect proportional to $I \cos \phi_0$ which changes the distribution of the flux but only decreases it slightly due to saturation. Thus even with non-inductive load the armature reaction causes a decrease in the flux crossing the air gap, and the e.m.f. E_1 generated by rotation is less than the no-load e.m.f. E_0. In addition, the armature reactance x and the resistance r both consume components of e.m.f. proportional to the current.

Therefore, at non-inductive load the terminal e.m.f. falls with increasing current as shown in curve (1), Fig. 246, which is the voltage characteristic for unity power factor.

With inductive load the demagnetizing effect is increased and the terminal e.m.f. falls off more, curve (2). With a capacity load in which the current leads the terminal e.m.f. the armature m.m.f. is magnetizing and so raises the terminal e.m.f., curve (3).

These voltage characteristics are calculated from equation 255 on page 263. The value of field current I_f is chosen and the corresponding value of E_0 obtained from Fig. 248. Any required power factor $\cos \phi$ is taken, the current I is varied and the values of E obtained and plotted as ordinates.

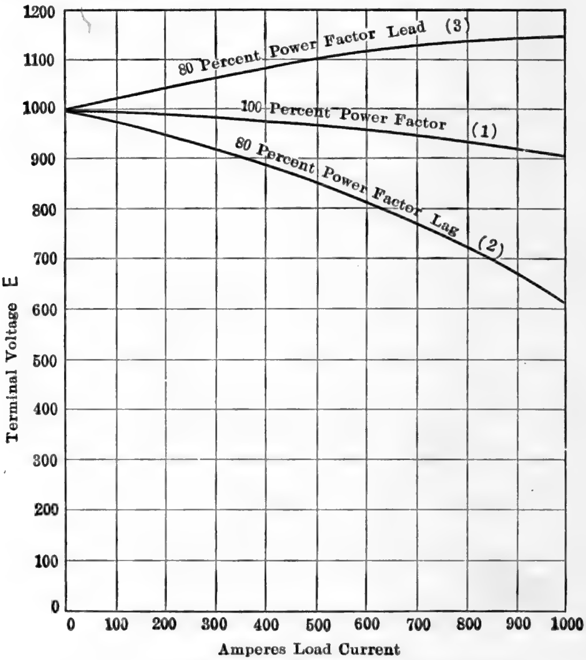


FIG. 246. Voltage characteristics of an alternator.

159. Compounding Curves. The "compounding curves" or "field characteristics" show the relation between the field current and armature current for a constant terminal e.m.f. at any required power factor.

Fig. 247 shows the compounding curves for unity power factor, curve (1), 80 per cent power factor lagging, curve (2), and 80 per cent power factor leading, curve (3).

At non-inductive load an increase of field current is required as the load current increases to maintain a constant terminal e.m.f.

With inductive load a much larger increase of field current is required to counteract the effect of the lagging current.

With capacity load the field current must be decreased in order to maintain a constant terminal e.m.f.

The same results are shown in the diagrams, Figs. 240, 241 and 242.

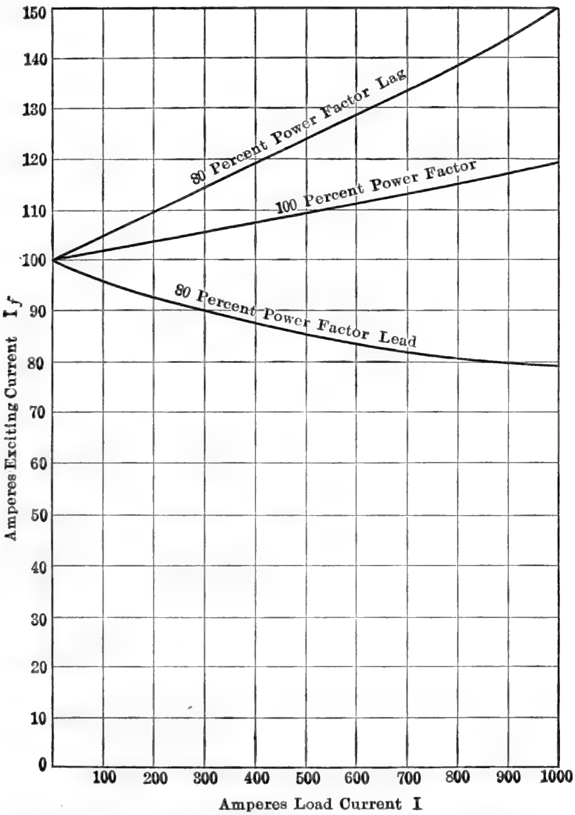


FIG. 247. Compounding curves of an alternator.

These compounding curves can also be calculated from equation

$$E_0 = \sqrt{(E \cos \phi + Ir)^2 + (E \sin \phi + Ix_0)}.$$

255, E remains constant, a certain value of $\cos \phi$ is chosen, I is varied and the value of E_0 is calculated. The corresponding value of I_f is obtained from the saturation curve, Fig. 248, and is plotted on the armature-current base.

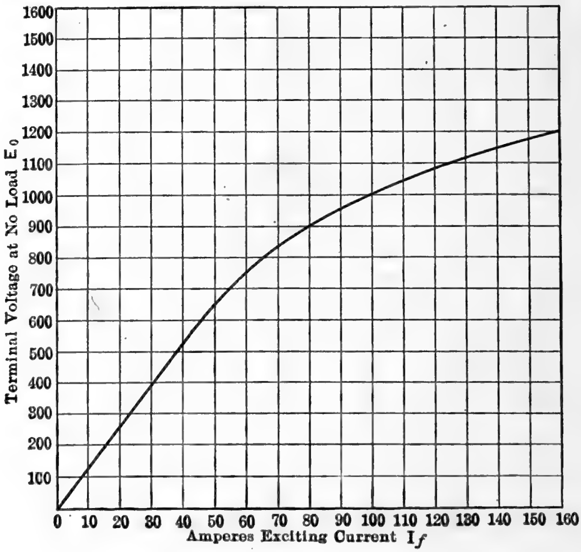


FIG. 248. Saturation curve of an alternator.

160. Short-circuit Currents of Alternators. If an alternator is short circuited, when operating with full-field excitation, the current is limited only by the synchronous impedance of the armature; it is

$$I_{sc} = \frac{E_0}{z_0} = \frac{E_0}{\sqrt{r^2 + (x_1 + x')^2}}, \quad \ldots \ldots \quad (258)$$

and will usually be from 3 to 6 times the normal full-load current.

At the instant of short circuit the current will be much larger than the value given by equation 258, because the component x' of the synchronous reactance, which represents the effect of armature reaction, does not act instantaneously to limit the current. It represents a change in the flux which interlinks with the field circuit of the machine and on account of the self-inductance of the field winding with its large number of turns, this change of flux cannot take place instantaneously but may take several seconds to become complete. During this time the short-circuit current is limited only by the true impedance of the armature, and is

$$I_{sc}' = \frac{E_0}{\sqrt{r^2 + x_1^2}}. \quad \ldots \ldots \ldots \quad (259)$$

It may reach 8 or 10 times full-load current.

The instantaneous short-circuit current is further increased due to the fact that with very heavy armature currents, the path of the true armature-reactance flux becomes very highly saturated and its permeability is therefore decreased. This results in a decrease of the reactance x_1.

161. Single-phase Short Circuits. In studying armature reaction in single-phase alternators it was shown that a double-frequency pulsation of the field is produced and a third harmonic of e.m.f. generated in the armature. Due to the high self-inductance of the field coils these effects are not very marked up to full load. At short circuit, however, the armature m.m.f. is very strong and produces a very large pulsation of the field. As a result large currents and e.m.f.'s are induced in the field winding.

The same effects are produced in the case of a short circuit of only one phase of a three-phase alternator; the armature reaction consists of a normal three-phase armature reaction with a single-phase reaction superimposed, of intensity corresponding to the increase of the short-circuit current over the normal current. A double-frequency pulsation of the field is produced and a large third harmonic of e.m.f. is generated in the two phases which are not short circuited.

162. Synchronous Motor. A synchronous motor is exactly the same as an alternator in construction and may be either single phase or polyphase. The single-phase motor is not self-starting and must be brought up to synchronous speed before being connected to the supply. It is, therefore, not used except in special cases. The polyphase motor when connected to the supply will accelerate and run up to synchronous speed but only a low voltage should be impressed on it at start or very large lagging currents will be drawn from the supply lines.

Fig. 249 (*a*) represents a two-phase, two-pole motor. The armature is stationary and is supplied with two-phase alternating currents, Fig. 249 (*b*). The armature m.m.f. is constant in value as in the alternator and revolves at synchronous speed in the anti-clockwise direction and produces a revolving field of constant value. Figs. 249 (*a*), (*c*) and (*d*) represent the armature m.m.f. at the instants (1), (2) and (3).

The speed of the field is directly proportional to the frequency

of the impressed e.m.f. and inversely proportional to the number of pairs of poles; it is

$$n = \frac{f}{\frac{p}{2}} = \frac{2f}{p} \text{ r.p.s.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (260)$$

This is the speed at which the motor operates and it is constant independent of the impressed e.m.f. of the field excitation and of the load.

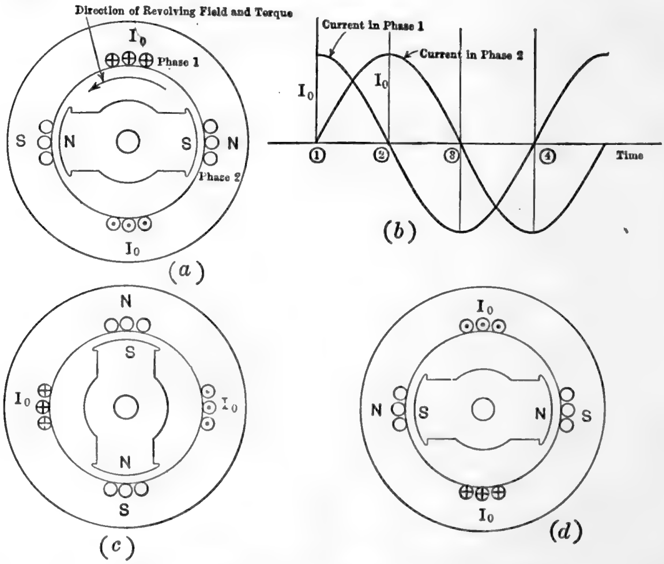


FIG. 249. Two-pole, two-phase synchronous motor.

163. Vector Diagrams. If an alternating e.m.f. E is impressed on the terminals of a synchronous motor of resistance r and synchronous reactance x_0 and a current I flows in the armature, the phase relation of the current and the impressed e.m.f. depends on the field excitation. In Fig. 250

$OE = \dot{E}$ = impressed e.m.f., which remains constant.

$OI = \dot{I}$ = armature current in phase with E.

$OE_1' = \dot{E}_1' = \dot{I}r$ = e.m.f. consumed by the resistance of the armature, in phase with I.

$OE_1'' = \dot{E}_1'' = j\dot{I}x_0$ = e.m.f. consumed by the synchronous reactance of the armature, 90 degrees ahead of I.

$OE_1 = \dot{E}_1 = \dot{E}_1' + \dot{E}_1'' = \dot{I}z$ = e.m.f. consumed by the synchronous impedance of the armature.

$OE_0' = \dot{E}_0'$ = e.m.f. generated in the armature by cutting the flux produced by the field m.m.f.

$OE_0 = \dot{E}_0$ = component of impressed e.m.f. required to overcome the generated e.m.f. $\dot{E}_0'$.

$OM_f = \dot{M}_f$ = field m.m.f., 90 degrees ahead of E_0'.

M_f is the value of field m.m.f. required to make the power factor of the motor unity.

Fig. 251 is the vector diagram for the motor when the current has the same value as before but lags behind the impressed e.m.f. by angle $\phi = 60°$.

Fig. 252 is the vector diagram when the current leads by angle $\phi = 60°$.

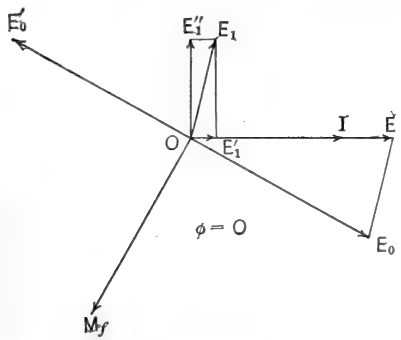


FIG. 250.

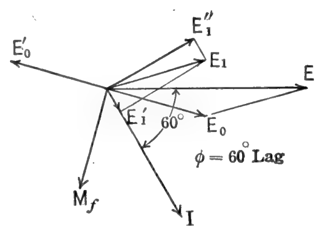


FIG. 251.

Referring to these diagrams it is seen that the field excitation required in a synchronous motor to produce a leading power factor or to cause the current to lead the impressed e.m.f. is greater than that required to produce a lagging power factor or to cause the current to lag behind the impressed e.m.f.

If, therefore, the field current of a synchronous motor is varied, there is no change in speed as in the direct-current motor, but the generated e.m.f. E_0' changes both its value and its phase relation with the impressed e.m.f. E and allows leading or lagging currents to flow to make up for the change in excitation; when the field current is decreased a component of current 90 degrees behind the impressed e.m.f. flows in the armature and magnetizes the field

and when the field current is increased a component of current 90 degrees ahead of the e.m.f. flows and demagnetizes the field.

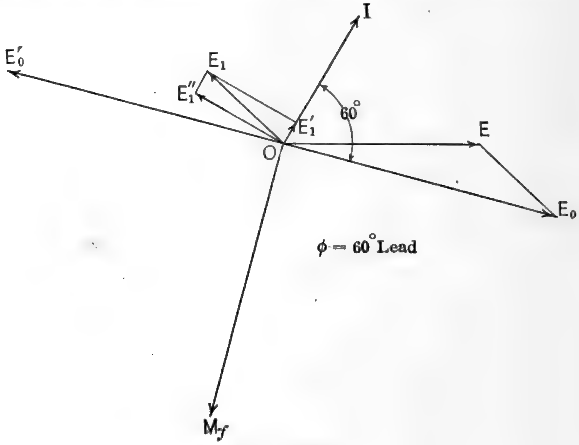


FIG. 252.

164. Characteristic Curves. The most important characteristic curves of the synchronous motor are (1) the compounding curves, or the relation between field current and armature current for given values of power factor, (2) the load characteristics showing the relation between armature current and output and power factor and output for a given value of field excitation and (3) the phase characteristics or V curves showing the relation between armature current and field current for given values of motor output.

To predetermine these curves it is necessary to know the resistance r and synchronous reactance x_0 of the armature and to have the no-load saturation curve showing the relation between the generated voltage E_0' and the field current I_f or field m.m.f. M_f.

The impressed e.m.f. E is constant.

165. Compounding Curves. The voltage equation of the synchronous motor is

$$\dot{E} = \dot{E}_0 + \dot{I}\dot{z}_0, \quad \ldots \ldots \quad (261)$$

or

$$\dot{E}_0 = \dot{E} - \dot{I}\dot{z}_0. \quad \ldots \ldots \quad (262)$$

If the current I is taken as the line of reference, the impressed e.m.f. can be expressed in rectangular coördinates as

$$\dot{E} = E \cos \phi + jE \sin \phi,$$

where ϕ is the angle of lag of the current.

The impedance drop in the armature is

$$\dot{E}_1 = \dot{I}\dot{z}_0 = Ir + jIx_0.$$

The generated e.m.f. is, therefore,

$$\dot{E}_0 = (E\cos\phi - Ir) + j(E\sin\phi - Ix_0) \quad . \quad . \quad (263)$$

and its absolute value is

$$E_0 = \sqrt{(E\cos\phi - Ir)^2 + (E\sin\phi - Ix_0)^2}. \quad . \quad . \quad (264)$$

This relation can also be obtained by reference to the vector diagram in Fig. 253,

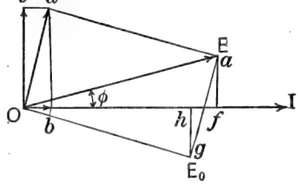


Fig. 253.

$$E_0 = \sqrt{\overline{oh}^2 + \overline{hg}^2}$$

but

$$oh = og - ob = E\cos\phi - Ir,$$

and

$$hg = oc - af = E\sin\phi - Ix_0,$$

therefore,

$$E_0 = \sqrt{(E\cos\phi - Ir)^2 + (E\sin\phi - Ix_0)^2}.$$

Fig. 254 represents the compounding curves for unity power factor, 80 per cent power factor leading and 80 per cent lagging.

To predetermine these curves the impressed e.m.f. E is maintained constant, a definite value of power factor is chosen for each curve, the armature current I is varied and the values of E_0 are calculated from equation 264. The values of field current I_f corresponding to the calculated values of E_0 are obtained from the saturation curve, Fig. 248, and are plotted as ordinates.

166. Load Characteristics. The power input to the motor armature is the product of the current and the in-phase component of impressed e.m.f.; it is

$$P_1 = EI\cos\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (265)$$

The electrical power transformed into mechanical power is the product of the current and the in-phase component of the generated e.m.f.; it is

$$P = I(E\cos\phi - Ir) = EI\cos\phi - I^2r, \quad . \quad . \quad (266)$$

and is less than the power input by the armature copper loss.

The power output is less than the mechanical power developed by the amount of the constant losses in the motor, namely, the iron friction and windage losses; the output, therefore, is

$$\begin{aligned} P_2 &= P - \text{constant losses} \\ &= P_1 - I^2r - \text{constant losses} \\ &= EI\cos\phi - I^2r - \text{constant losses}. \quad . \quad . \quad (267) \end{aligned}$$

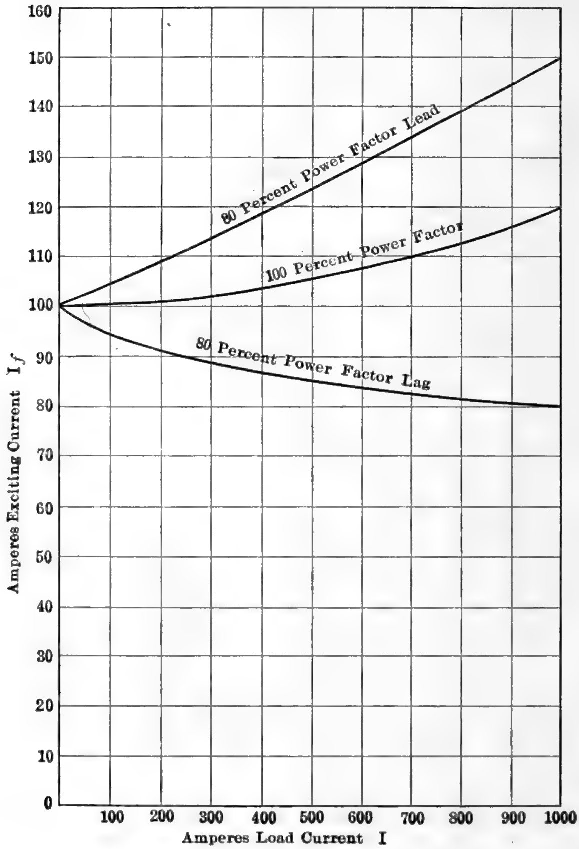


FIG. 254. Compounding curves of a synchronous motor.

Fig. 255 represents the load characteristics for a given value of field current I_f. Since I_f is constant E_0 is constant. I is varied and the corresponding values of $\cos \phi$ are obtained from equation 264. These values are substituted in equation 267 and the values of I and $\cos \phi$ are plotted on a base of power output.

At light loads the power factor is low and leading and the motor is over excited; as the load is increased the power factor increases until it reaches 100 per cent at a value of load depending on the field excitation; beyond this point the power factor decreases again and becomes lagging. The current increases continually but is finally limited by the synchronous impedance of the armature.

The power output cannot increase indefinitely but reaches a maximum when the decrease in power factor overcomes the increase in current and then the motor becomes unstable and falls out of synchronism and stops.

The maximum power input occurs when

$$P_1 = EI \cos \phi \text{ is maximum.}$$

The maximum power output occurs when

$$P_2 = EI \cos \phi - I^2r - \text{constant losses is maximum.}$$

These maximum values of input and output are far beyond the heating limits of the motor.

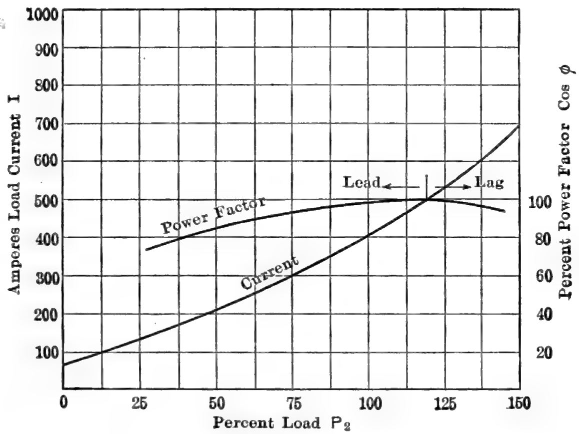


FIG. 255. Load characteristics of a synchronous motor.

The torque developed by the motor at any output is directly proportional to the output. If T is the torque in foot pounds developed in the armature, the mechanical power developed is

$$P = \frac{2\pi NT}{33{,}000} \times 746 \text{ watts} = EI \cos \phi - I^2r,$$

where N is the synchronous speed in revs. per min.; thus

$$T = \frac{EI \cos \phi - I^2r}{2\pi N} \cdot \frac{33{,}000}{746} \text{ ft. lbs.}$$

and the torque available is

$$T_2 = \frac{EI \cos \phi - I^2r - \text{constant losses}}{2\pi N} \times \frac{33{,}000}{746}.$$

167. Phase Characteristics. If the field excitation of a motor with constant output is varied, the armature current changes both its value and its phase relation with the impressed e.m.f. For each output there is a certain value of field excitation which makes the current a minimum and brings it in phase with the e.m.f.; as the excitation is decreased below this value the current increases and becomes lagging; as the excitation is increased the current increases and becomes leading.

In Fig. 256 are shown the phase characteristics for outputs, $P_2 = 0$ or at no load, $P_2 =$ half load, $P_2 =$ full load and $P_2 = 150$ per cent load.

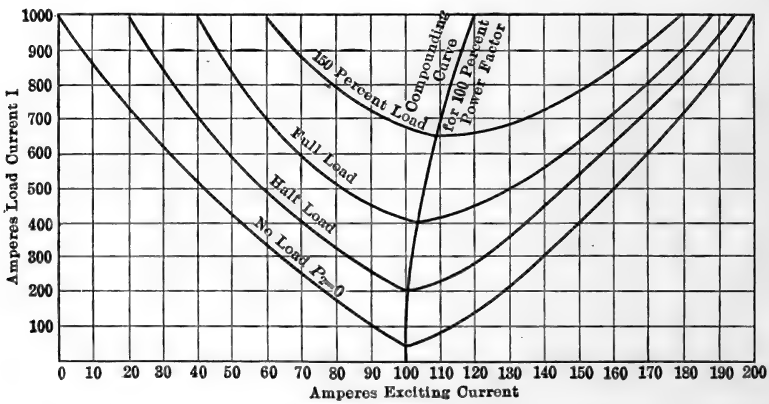


FIG. 256. Phase characteristics or "V" curves of a synchronous motor.

For each curve the output $P_2 = EI \cos \phi - I^2r -$ constant losses is kept constant; thus,

$$\cos \phi = \frac{P_2 + I^2r + \text{constant losses}}{EI} \quad . \quad . \quad . \quad (268)$$

and
$$E_0 = \sqrt{(E \cos \phi - Ir)^2 + (E \sin \phi - Ir_0)^2}.$$

As I varies the corresponding value of $\cos \phi$ is found from equation 268 and by substituting I, $\cos \phi$ and $\sin \phi$ in equation 264, the values of E_0 are found. These are replaced by the corresponding values of field current I_f obtained from the no-load saturation curve.

The lowest point on each curve represents the smallest current input for the given output and thus represents the condition of unity power factor. The curve joining these lowest points is the

compounding curve for unity power factor. If the phase characteristics are very steep a slight change in field excitation produces a large change in armature current, or a large component of wattless current is required to correct for a slight variation in field excitation. This is the case in a motor with small synchronous reactance or small armature reaction and the motor is unstable. If the synchronous reactance is large only a slight change in armature current is produced by a change in field excitation and the phase characteristics are flat and the motor is stable.

168. Synchronous Compensators. Since by varying the field excitation of a synchronous motor the power factor can be made either leading or lagging, such machines can be used to improve the power factor of transmission lines or distributing circuits by drawing wattless leading currents to compensate for the wattless lagging currents required by the load. The fields must be over excited and the synchronous reactance should not be very large. This is one of the most important characteristics of the synchronous motor and is being applied to an ever-increasing extent. The synchronous compensator usually operates without load drawing the required wattless leading current and a small power current to supply its own losses. In some cases, however, it may be advantageous to supply some load from it.

169. Starting. In order to improve the starting torque of synchronous motors and also to prevent hunting short-circuited grids are placed in the pole faces, Fig. 257, or between the poles, and in addition the poles are sometimes made solid. The field winding at start may either be open or else short circuited. In the following discussion it will be assumed that the field circuit is open. Torque is produced in two distinct ways.

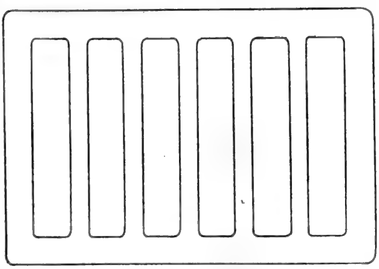


Fig. 257. Copper grid in a pole face.

(1) The revolving field sweeps across the grids and the solid pole faces and generates e.m.f.'s and currents in them. These currents react on the field and produce torque which makes the rotor follow the field. The rotor can never be brought up to synchronous speed by this torque because the e.m.f.'s and currents are only induced below synchronous speed.

(2) The second way in which torque is produced can be understood by reference to Fig. 258.

The current in phase 1 grows to a maximum and produces a north pole on the armature at N and a south pole on the rotor at S. As the north armature pole moves around to the left the south pole on the field becomes weaker but it cannot immediately disappear since eddy currents are induced in the solid field poles which retard the decrease of flux. The armature pole exerts a pull on the residual field pole and produces torque. The revolving north pole is succeeded by a south pole which induces a north pole on the rotor and exerts a torque on it. The torque produced in this way combines with the torque produced by the currents induced in the pole faces and grids and brings the motor up nearly to

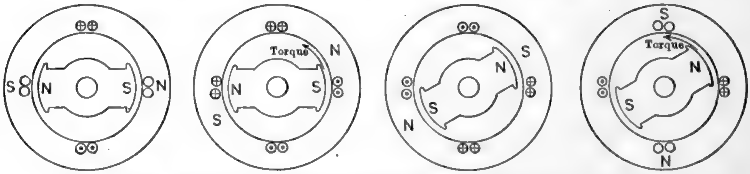


Fig. 258. Starting torque of a polyphase synchronous motor.

synchronous speed. When the armature pole is moving very slowly across the field pole the two lock together in the position of minimum reluctance and the motor runs at synchronous speed. The field circuit can then be closed and the impressed e.m.f. raised to its full value. When starting in this way the motor draws a very large lagging current since the impressed voltage is consumed by the synchronous impedance of the armature, and the power factor is very low. The impressed voltage at start must be reduced to about one third of its full value in order to reduce the starting current. It is the flux of armature reaction which produces the starting torque and, therefore, a motor with high armature reaction will give better starting torque than one of low armature reaction.

When the motor is running below synchronous speed large voltages sometimes reaching 5000 volts or more are induced in the open-field winding by the revolving armature flux. Such voltages are dangerous and may puncture the insulation of the field or endanger the lives of operators. Both the magnitude and frequency of these induced voltages become zero when the motor reaches synchronous speed.

If the field winding is short circuited while coming up to synchronism these large induced voltages will not exist but the starting torque of the motor will in general be reduced. The short-circuited field winding tends to act in somewhat the same way as the grids but its inductance is so high that the flux produced through it by the armature m.m.f. is small and this reduction of the flux in the field poles more than counterbalances the torque due to induced currents. In the majority of cases, however, it is better to start the motor with the field winding short circuited since the safety of operation is thereby increased. When synchronous speed is reached the full excitation should be applied and then the impressed voltage raised to full value.

170. Parallel Operation of Alternators. Before an alternator is connected in parallel with another machine which is supplying power, the incoming machine must be adjusted to give the same voltage, must have the same frequency and must be in phase.

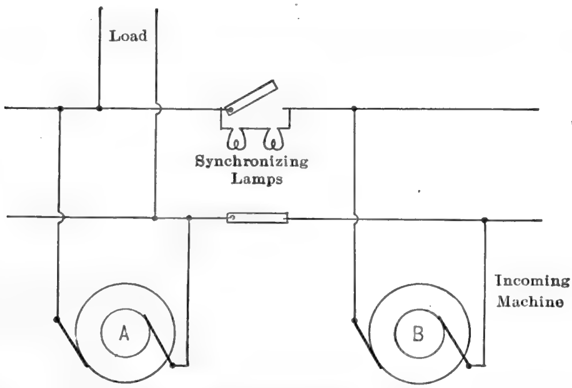


FIG. 259. Parallel operation of alternators.

The condition of synchronism may be indicated by incandescent lamps or by some form of synchroscope, Art. 176.

In Fig. 259 when the incoming machine B is in synchronism with A there is no voltage across the switch and the lamps are dark. The two machines are assumed to have been adjusted to give equal voltages.

If the frequency of B is lower or higher than that of A, there will be a slow pulsation of the light showing the difference between the frequencies. When the pulsations are very slow and the periods of darkness long the switch may be closed and the field of B adjusted

until it takes its proper share of the load. Lamps are not very satisfactory since they do not show whether the incoming machine is running too fast or too slow.

171. Effect of Inequality of Terminal Voltage. If two alternators have the same frequency and are in phase but have not their fields adjusted to give the same terminal voltage, a wattless current will flow between the two machines leading and magnetizing in the machine of lower field excitation and lagging and demagnetizing in the machine of higher field excitation.

If E_2 the voltage of B is lower than E_1 the voltage of A then E' the difference of the two will act in the local circuit through the two armatures in series and will produce a current I' lagging nearly 90 degrees behind E' and E_1 and leading E_2, Fig. 260.

The circulating current is

$$I' = \frac{E'}{z_1 + z_2}, \qquad (269)$$

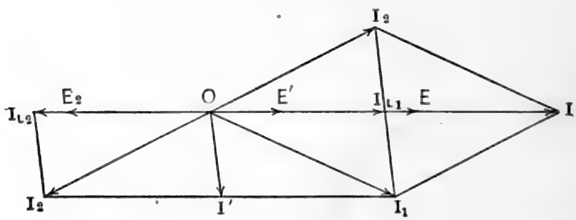


FIG. 260.

where $z_1 = r_1 + jx_1$ is the synchronous impedance of A and $z_2 = r_2 + jx_2$ is the synchronous impedance of B.

This current lowers the terminal e.m.f. of A since it is lagging in A and raises the terminal e.m.f of B since it is leading in B.

The total current in A will lag behind the current in the load circuit and the current in B will lead the current in the load circuit. In Fig. 260

$OE_1 = E_1$ is the terminal e.m.f. of A before synchronizing.
$OE_2 = E_2$ is the terminal e.m.f. of B before synchronizing.
$OE' = E'$ is the e.m.f. which produces the circulating current.
$OI' = I'$ is the circulating current.
$OI_{L_1} = I_{L_1}$ is the load current of A.
$OI_{L_2} = I_{L_2}$ is the load current of B.
$OI_1 = I_1$ is the total current of A.
$OI_2 = I_2$ is the total current of B.
$OI = I = I_{L_1} + I_{L_2}$ is the load current.

By adjusting the field rheostats the wattless circulating currents can be eliminated for any load, but if the two machines have different voltage characteristics, as the load varies wattless currents will circulate to correct for the differences of excitation.

With machines of reasonably high armature reaction the wattless cross currents are small even with large variations of excitation.

172. Effect of Inequality of Frequency. Two alternators operating in parallel must have the same average frequency, but one may instantaneously drop behind or run ahead of the other.

Alternators driven by turbines or electric motors will have a constant angular velocity but when the prime movers are steam engines or gas engines the angular velocity will pulsate about its average value during each revolution.

If two machines in parallel are excited to give the same terminal voltage and one falls behind the other, a power cross current will circulate through the armatures and transfer energy from the leading to the lagging machine.

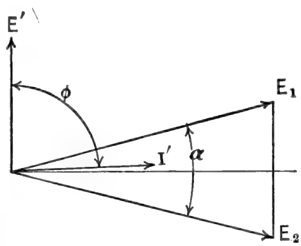


FIG. 261.

Fig. 261 shows the case of two machines when E_2 lags behind E_1 by angle α. The e.m.f. producing the circulating current is the vector difference between E_1 and E_2 and it is

$$E' = 2\,E_1 \sin\frac{\alpha}{2}. \qquad (270)$$

The circulating current is $\underline{I}' = \dfrac{E'}{\underline{z}_1 + \underline{z}_2}$, its absolute value is

$$I' = \frac{2\,E_1 \sin\frac{\alpha}{2}}{\sqrt{(r_1 + r_2)^2 + (x_1 + x_2)^2}} \qquad (271)$$

and it lags behind E' by angle

$$\phi = \tan^{-1}\frac{x_1 + x_2}{r_1 + r_2},$$

which is very nearly 90 degrees.

The current is therefore nearly in phase with the terminal voltage E_1 and in phase opposition to the terminal voltage E_2; it thus transfers power from the leading to the lagging machine.

The component of I' in phase with E', that is, the power component does not represent a power transfer, but the reactive component, which is in quadrature behind E', transfers the power required to keep the machines in step. This lagging component of current is proportional to the reactance of the two armatures and therefore a fairly high synchronous reactance or armature reaction is required in alternators and synchronous motors to give a good synchronizing power. If the synchronous reactances are too large the circulating current will be limited and the synchronizing power reduced.

Take the case where one machine falls behind the other in phase by angle $\alpha = 20°$; the circulating current is

$$I' = \frac{2\,E_1 \sin 10°}{\sqrt{(r_1 + r_2)^2 + (x_1 + x_2)^2}} = 0.168 \frac{2\,E_1}{\sqrt{(r_1 + r_2)^2 + (x_1 + x_2)^2}},$$

but $\dfrac{2\,E_1}{\sqrt{(r_1 + r_2)^2 + (x_1 + x_2)^2}}$ is the average short-circuit current of the two alternators with full-field excitation, and this is about six times the average full-load current; thus the circulating current is

$$\begin{aligned} I' &= 0.168 \times 6 \times \text{full-load current} \\ &= \text{full-load current, approximately.} \end{aligned}$$

These power cross currents, if of great magnitude, tend to tear the machines out of synchronism and also produce fluctuations of the voltage.

If the speed characteristics of the prime movers are not the same and the speed of one machine tends to fall below the other as the load on the system is increased, then the machine driven by the prime mover of closer speed regulation takes more than its share of the load and so relieves the other machine and keeps its speed up.

Thus to insure a proper division of the load between alternators operating in parallel, it is necessary that their prime movers shall have similar speed characteristics, that is, that their speed shall fall under load by the same amount and in the same manner. It is therefore preferable that the prime movers should have drooping speed characteristics.

The voltage characteristics of the alternators have no effect on the division of the load, but they do affect the amount of the wattless cross currents between the machines.

173. Effect of Difference of Wave Form. If two machines in parallel are adjusted to give the same effective value of voltage

but have different wave shapes, then, since, due to the presence of the higher harmonics, the voltages are not equal at every instant, wattless cross currents will flow to correct these inequalities in voltage. These currents will usually be very small since the voltages producing them are small and they are of high frequency and thus the path through the two machines offers a high impedance to them. The impedance is, however, only the true impedance and not the synchronous impedance.

Thus three kinds of cross currents may exist in parallel operation of alternators, (1) wattless currents transferring magnetization between the machines due to a difference of terminal voltage, (2) currents transferring power between the machines due to phase displacements between their voltages, and (3) higher frequency wattless currents due to differences of wave form.

174. Hunting. If two alternators are operating in parallel and one drops behind the other in phase due to a sudden decrease in the speed of its prime mover, the second machine supplies power to pull it into synchronism again. The impulse received causes it to swing past its mean position and it oscillates a few times before falling into step. The period of the oscillation depends on the weight of the rotating mass and on the strength of the magnetic field, that is, on the pull between the field poles and the induced armature poles. The greater the weight of the machine the lower will be the frequency, and the stronger the field excitation the greater will be the magnetic pull and the higher will be the frequency.

If the action producing the speed pulsation is repeated periodically and coincides with the natural period of the machine the oscillation instead of dying out will increase in amplitude until it is limited by the friction losses produced or until the machines fall out of step. When the oscillations tend to become cumulative the machines are said to be hunting.

Hunting may occur in a similar way in the case of a synchronous motor supplied by an alternator. If the load on the motor suddenly increases it falls back in phase to receive the extra power required and oscillates about its final phase position before running again in synchronism. This oscillation may become continuous as in the case of alternators.

Hunting is reduced by putting short-circuited grids in the pole faces or between the poles or in some cases by placing a complete

squirrel-cage winding in the pole faces. At synchronous speed the armature-reaction flux is stationary relative to the fields and, therefore, does not produce any current in the grids but if the machine falls below or runs above synchronous speed, the flux sweeps across the grids and produces e.m.fs. in them and large currents flow which react on the field and tend to hold the machine exactly in synchronism.

Hunting may sometimes be reduced or eliminated by changing the field excitation and thus changing the natural period of oscillation of the machine.

Maohines with high armature reaction are much less liable to hunt than machines with low armature reaction since the high armature reaction reduces the circulating currents produced by changes in phase.

175. Automatic Voltage Regulator for Alternating-current Generators. The Tirrill regulator described in Art. 120 and manufactured by the General Electric Co., may, with slight changes, be applied to regulate the voltage of alternators. The desired voltage is maintained by opening and closing a short circuit across the exciter field rheostat.

The method of operation of the regulator can be understood from the diagram of connections shown in Fig. 262. The direct-current control magnet is connected to the exciter bus bars and has a fixed core in the bottom and a movable core in the top attached to a pivoted lever, at the other end of which is a spring which closes the main contacts. The alternating-current control magnet has a potential winding connected across one phase of the alternator and it may also have a compensating winding connected through a current transformer to one of the feeders. The core is movable and is connected to a pivoted lever controlled by a counterweight. The relay magnet is differentially wound and is connected as shown.

Operation. The direct- and alternating-current control magnets are adjusted for the required voltage by means of the counterweight. The exciter field rheostat is then set to reduce the voltage about 65 per cent below normal. This weakens both of the control magnets and the spring closes the main contacts and demagnetizes the relay magnet. The pivoted armature is released and the relay contacts are closed and thus short circuit the exciter field rheostat and immediately raise the exciter voltage and the alternator voltage. When the alternator voltage reaches the value for

which the regulator is adjusted, the main contacts open again, the relay magnet is again magnetized and the short-circuit on the exciter field rheostat is opened. This reduces the voltage as before and the cycle of operations is repeated at a very rapid rate and maintains a constant voltage at the terminals of the alternator if the compensating winding is not connected.

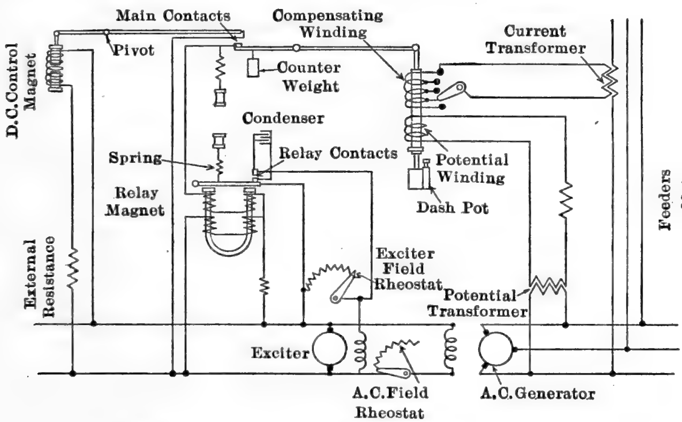


FIG. 262. Automatic voltage regulator for alternators.

When it is necessary to maintain a constant voltage at the receiver end of the line the compensating winding is connected as shown. As the load increases it brings the main contacts closer together and so increases the time of short circuit of the field rheostat and thus increases the terminal voltage of the alternator. Using a current transformer and a dial switch any line drop up to 15 per cent can be compensated for, but only at a given power factor. When the power factor of the load varies through a wide range a line compensator, Fig. 263(*a*), should be used in conjunction with the potential coil and the compensating coil should be disconnected.

The line compensator forms a local circuit with the same voltage characteristics as the main line. The shunt transformer T_1 gives a secondary voltage proportional to the generator voltage. The current transformer T_2 produces through the circuit rx a current proportional to the load current. r is a resistance which consumes a voltage proportional to and in phase with the resistance drop in the line and this voltage is transferred to the compensator circuit by the potential transformer T_3. x is a reactance and consumes

a voltage proportional to the reactance drop in the line. This voltage is transferred to the compensator circuit by the transformer T_4 which also forms the reactance. Thus there are in the compensator circuit three voltages proportional respectively to the generator voltage and the resistance and reactance drops. If the same proportions are maintained in each case, the voltage between the terminals AB will be proportional to the voltage at the end of the line, and, therefore, if the potential coil of the regulator is connected across AB it will maintain a constant voltage at the receiver end of the line. In the case of transmission lines of very high voltage a correction must be made for the capacity of the lines. The compensator shown is arranged for a single-phase circuit. With a three-phase alternator, as in Fig. 262, two current transformers must be used connected as shown in Fig. 263(*b*).

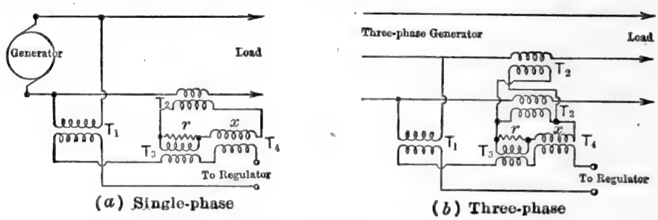


FIG. 263. Line compensator.

One automatic voltage regulator can be applied to control the voltage of a system operating two or more alternators in parallel.

The regulator may also be applied to the exciter of a synchronous compensator to maintain a constant power factor at the receiver end of the line.

176. Synchroscope. A synchroscope is an instrument which indicates, (1) whether the incoming machine is running too fast or too slow and (2) the exact instant when synchronism is reached.

One form of synchroscope is shown in Fig. 264. It has a laminated bipolar magnetic circuit M excited by field coils FF, which are connected across the alternating-current bus bars at (1) and (2) and produce an alternating field. The core C is also laminated and carries two windings A and B at right angles to one another. Their common terminal is connected to one side of the incoming machine at (4). The other terminals of A and B are connected through a resistance r and a reactance x respectively to the other side of the machine at (3).

The current in F is in phase with the line voltage, the current in A is in phase with the machine voltage and the current in B is in quadrature behind the machine voltage.

When the incoming machine is exactly in synchronism the coil A takes the position shown in the figure, since the current in A is in phase with the current in F. When the machine is 90 degrees behind or ahead of the position of synchronism, the current in B is in phase with the current in F and the armature turns through 90 degrees and brings the coil B in line with the poles.

For intermediate phase relations the armature takes intermediate positions, such that the revolving field produced by the

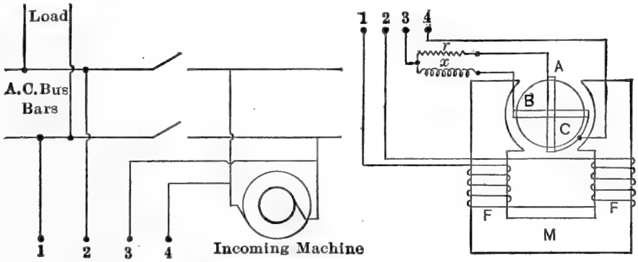


FIG. 264. Synchroscope.

armature winding is in line with the field poles when the current in F is maximum. The phase relation is indicated by a pointer on the dial of the synchroscope.

When the frequency of the incoming machine is lower than that of the line, the phase of the current in A continually falls behind that of F and the pointer rotates in the direction marked "Slow." When the incoming machine is running too fast, the rotation of the pointer is in the opposite direction marked "Fast."

When the machine is running exactly at synchronous speed and is exactly in phase, the pointer is vertical and stationary. The switch can then be closed and the excitation adjusted until the machine takes its proper share of the load.

The synchroscope described is for a single-phase circuit. It can be used for a two-phase machine by connecting the two coils A and B to the two phases of the machine and the coil F across one phase of the line.

For three-phase systems the armature is wound with a three-phase winding connected to the three phases of the machine.

CHAPTER VI

TRANSFORMERS

177. The Constant-potential Transformer. The constant-potential transformer consists of one magnetic circuit interlinked with two electric circuits, the primary circuit which receives energy and the secondary circuit which delivers energy. Its function is to transform electric power from low voltage and large current to high voltage and small current, or the reverse. In step-up transformers the primary is the low-voltage (L.V.) side and the secondary is the high-voltage (H.V.) side. In step-down transformers the primary is the high-voltage side.

In the following discussion letters with the subscript 1 will be used to represent primary quantities and with the subscript 2 to represent secondary quantities.

Fig. 265 represents a transformer. The core is made up of thin sheets of iron or steel of high permeability with small hysteresis and eddy current loss.

The primary winding consists of n_1 turns in series and has a resistance of r_1 ohms, a self-inductive or leakage reactance of x_1 ohms and thus an impedance of $z_1 = r_1 + jx_1$ ohms.

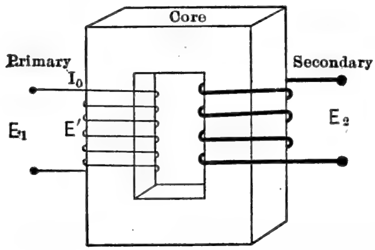
The secondary winding consists of n_2 turns in series. Its resistance is r_2 ohms, its reactance is x_2 ohms and its impedance is $z_2 = r_2 + jx_2$ ohms.

When an alternating e.m.f. E_1 is impressed on the primary winding with the secondary open, a current I_0 flows in the primary and produces an alternating flux through the core of maximum value Φ. The current I_0 is called the exciting current of the transformer and consists of two components (Fig. 266) I_M in phase with the flux Φ, called the magnetizing current, and I_c in quadrature ahead of the flux and in phase with the impressed e.m.f., called the core-loss current. The product of E_1 and I_c is the power wasted in the core loss, that is, in supplying the hysteresis and eddy current losses of the transformer. The exciting current, therefore, lags by an angle θ_0, which is less than 90 degrees, behind the impressed e.m.f. Cos θ_0 is the no-load power factor of the transformer.

The exciting current I_0 is of the order of 10 per cent of full-load current and the no-load power factor is of the order of 30 per cent.

The alternating flux produced by the magnetizing current links with the secondary winding and induces in it an e.m.f.,

$$e_2 = n \frac{d\phi}{dt} 10^{-8} \text{ volts.}$$



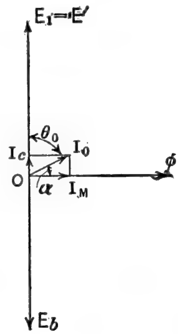


FIG. 265. Constant potential transformer. FIG. 266.

If the frequency is f cycles per second and the flux follows a sine wave of maximum value Φ, the instantaneous e.m.f. induced in the secondary is

$$\begin{aligned} e_2 &= n_2 \frac{d}{dt}(\Phi \sin 2\,\pi f t)\, 10^{-8} \text{ volts} \\ &= 2\,\pi f n_2 \Phi \cos 2\,\pi f t\, 10^{-8} \text{ volts} \\ &= 2\,\pi f n_2 \Phi \sin (2\,\pi f t - 90)\, 10^{-8}; \qquad (272) \end{aligned}$$

this is a sine wave of e.m.f. in quadrature behind the flux, of maximum value

$$E_{2\max} = 2\,\pi f n_2 \Phi\, 10^{-8} \qquad (273)$$

and effective value

$$E_2 = \frac{2}{\sqrt{2}} \pi f n_2 \Phi\, 10^{-8} = 4.44\, f n_2 \Phi\, 10^{-8} \text{ volts.}$$

The flux also links with the primary winding and induces in it an e.m.f. of instantaneous value

$$\begin{aligned} e_b &= n_1 \frac{d\phi}{dt} 10^{-8} \\ &= 2\,\pi f n_1 \Phi\, 10^{-8} \sin (2\,\pi f t - 90), \end{aligned}$$

a sine wave of maximum value

$$E_{b\,\max} = 2\,\pi f n_1 \Phi\, 10^{-8}$$

and effective value

$$E_b = 4.44\, f n_1 \Phi\, 10^{-8} \text{ volts.} \qquad (274)$$

This e.m.f. induced in the primary is almost equal in value and opposite in phase to the impressed e.m.f., the vector sum of the two being the small component of impressed e.m.f. required to drive the exciting current through the impedance of the primary winding. Thus

$$\underset{\cdot}{E}_1 + \underset{\cdot}{E}_b = \underset{\cdot}{I}_0\underset{\cdot}{z}_1.$$

This component has been neglected in Fig. 266. The induced e.m.fs. E_b and E_2 are directly in phase since they are produced by the same flux, and their intensities are in the ratio of the turns on the two windings; therefore,

$$\frac{E_b}{E_2} = \frac{4.44\,fn_1\Phi\,10^{-8}}{4.44\,fn_2\Phi\,10^{-8}} = \frac{n_1}{n_2} = \text{ratio of turns.} \quad . \quad . \quad (275)$$

If the secondary is connected to a receiver circuit of impedance $\underset{\cdot}{Z} = R + jX$, a current I_2 flows in it. The primary current is at the same time increased by a component I', the primary load current, which exerts a m.m.f. equal and opposite to that of the secondary current.

Thus

$$n_1I' = n_2I_2$$

and

$$\frac{I_2}{I'} = \frac{n_1}{n_2} = \text{ratio of transformation.} \quad . \quad . \quad . \quad (276)$$

The resultant m.m.f. acting on the magnetic circuit of the transformer is still that of the primary exciting current and the flux threading the two windings remains almost constant.

The primary current under load is I_1 and has two components I_0 the exciting current, which is proportional to the flux, and I' the load current which is proportional to the secondary current.

The exciting current $\underset{\cdot}{I}_0$ can be expressed as the product of the primary induced e.m.f. E_b and the primary exciting admittance $\underset{\cdot}{y}_0 = g_0 - jb_0$; thus

$$\underset{\cdot}{I}_0 = \underset{\cdot}{E}_b\,(g_0 - jb_0) = \underset{\cdot}{E}'\,(g_0 - jb_0), \quad . \quad . \quad . \quad . \quad (277)$$

where E' is the component of impressed e.m.f. required to overcome the induced e.m.f. E_b.

The primary load current is $I' = \frac{n_2}{n_1} I_2$, and is opposite in phase to I_2.

As the load on the transformer is increased, the primary induced e.m.f. decreases (except when the power factor of the load

is leading) because a larger component of impressed e.m.f. is consumed in driving the current through the primary impedance, thus,

$$E' = -E_b = E_1 - I_1 z_1. \quad \ldots \ldots \quad (278)$$

A smaller flux, is, therefore, required and a smaller exciting current. The decrease in flux from no-load to full-load non-inductive is 1 or 2 per cent and for an inductive load of 50 per cent power factor is only 5 or 6 per cent. With anti-inductive load the flux increases.

The secondary induced e.m.f., which is proportional to the primary, decreases with it.

The secondary terminal e.m.f. E is less than the secondary induced e.m.f. by the e.m.f. consumed by the secondary impedance; thus,

$$E = E_2 - I_2 z_2. \quad \ldots \ldots \quad (279)$$

178. Vector Diagrams for the Transformer. Fig. 267 is the vector diagram of e.m.f.'s and currents in a transformer with a load impedance $Z = R + jX$ and a load power factor

$$\cos\theta = \frac{R}{\sqrt{R^2 + X^2}}.$$

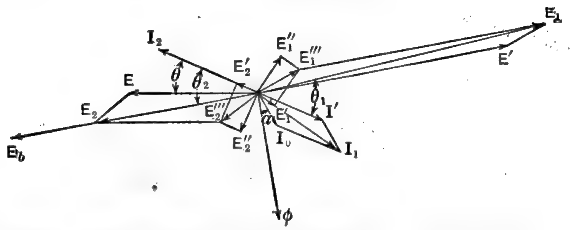


FIG. 267. Vector diagram of a transformer with inductive load.

$OE = E$ = secondary terminal e.m.f.

$OI_2 = I_2$ = secondary load current lagging behind E by angle θ.

$OE_2' = E_2' = I_2 r_2$ = e.m.f. consumed by secondary resistance, in phase with I_2.

$OE_2'' = E_2'' = jI_2 x_2$ = e.m.f. consumed by secondary reactance, in quadrature ahead of I_2.

$OE_2''' = E_2''' = I_2 z_2$ = e.m.f. consumed by secondary impedance.

$OE_2 = E_2 = E + E_2'''$ = e.m.f. induced in the secondary winding by the alternating flux Φ.

$OE_b = E_b$ = e.m.f. induced in the primary winding by the flux Φ $E_b = \frac{n_1}{n_2} E_2$.

$O\Phi \quad = \underset{\cdot}{\Phi} \quad$ = flux threading both primary and secondary windings in quadrature ahead of $\underset{\cdot}{E}_b$ and $\underset{\cdot}{E}_2$.

$OI_0 \quad = \underset{\cdot}{I}_0 \quad$ = primary exciting current leading the flux Φ by an angle $90 - \theta_0$, where θ_0 is the primary power factor at no load.

$OI' \quad = \underset{\cdot}{I}' \quad$ = primary load current in phase opposition to $\underset{\cdot}{I}_2$,

$$\underset{\cdot}{I}' = \frac{n_2}{n_1}\underset{\cdot}{I}_2.$$

$OI_1 \quad = \underset{\cdot}{I}_1 \quad$ = total primary current.

$OE' \quad = \underset{\cdot}{E}' \quad$ = component of primary impressed e.m.f. required to overcome the primary induced e.m.f. E_b.

$OE_1' \quad = \underset{\cdot}{E}_1' \quad = \underset{\cdot}{I}_1 r_1$ = e.m.f. consumed by primary resistance in phase with $\underset{\cdot}{I}_1$.

$OE_1'' = \underset{\cdot}{E}_1'' = j\underset{\cdot}{I}_1 x_1$ = e.m.f. consumed by primary reactance in quadrature ahead of $\underset{\cdot}{I}_1$.

$OE_1''' = \underset{\cdot}{E}_1''' = \underset{\cdot}{I}_1 z_1$ = e.m.f. consumed by primary impedance.

$OE_1 \quad = \underset{\cdot}{E}_1 \quad = \underset{\cdot}{E}' + \underset{\cdot}{E}_1'''$ = primary impressed e.m.f.

$\theta_1 \quad$ = angle of lag of the primary current behind the impressed e.m.f. $\cos\theta_1$ = primary power factor.

$\theta_2 \quad$ = angle of lag of the secondary current behind the secondary induced e.m.f.

Fig. 268 shows the vector diagram of a transformer with a non-inductive load and Fig. 269 with a capacity load of 50 per cent power factor leading.

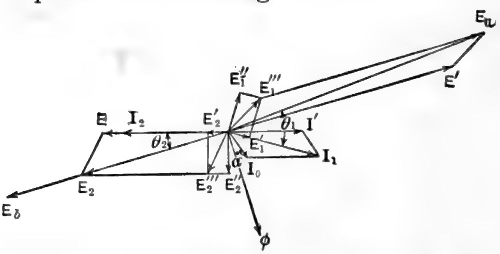


FIG. 268. Vector diagram of a transformer with a non-inductive load.

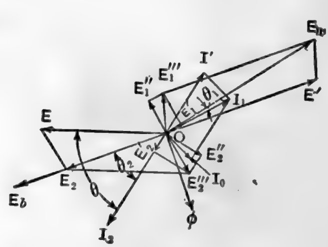


FIG. 269. Vector diagram of a transformer with a load power factor of 50 per cent leading.

179. Exciting Current. When a sine wave of e.m.f. is impressed on the primary winding of a transformer, a sine wave of flux must be produced linking with the primary winding. The exciting current which produces th[illegible]x cannot be a sine wave

on account of the lag of flux due to hysteresis. This is shown in Fig. 270. Curve (1) *abcd* is a hysteresis loop for the transformer iron, plotted with values of flux as ordinates on a base of exciting current. Curve (2) is the sine wave of flux in the core and curve (3) is the wave of exciting current. The method of obtaining curve (3) can be seen from the figure. The maximum values of flux and current must occur together; when the flux is zero the current has a value *oa* and when the current is zero the flux has a negative value *og*.

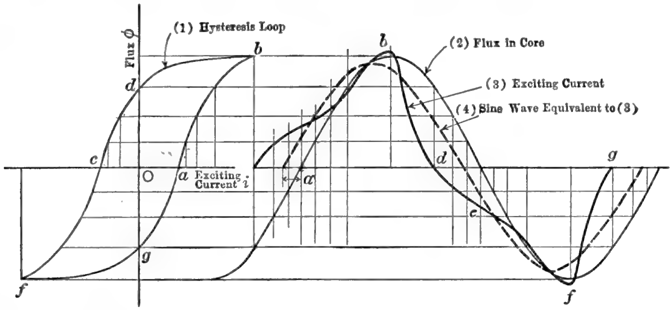


FIG. 270. Exciting current.

For purposes of analysis the current wave (3) is replaced by the equivalent sine wave (4). The current wave (4) leads the flux wave (2) by an angle α, which is called the angle of hysteretic advance. If the eddy current loss is small enough to be neglected, $\alpha = 90 - \theta_0$, where $\cos \theta_0$ is the no-load power factor.

180. Leakage Reactances. Figs. 271 and 272 show the leakage paths around the windings of a "shell" type and "core" type transformer. Since the low-voltage windings are placed next to the iron, the leakage path surrounding the low-voltage winding is of slightly lower reluctance than that surrounding the high-voltage winding and the reactance is correspondingly larger.

In the shell-type transformer the two windings are divided into a number of sections and high-voltage and low-voltage coils placed alternately to reduce the reactances.

The reactance voltage of a transformer with full-load current is about 10 per cent of the total voltage. If, therefore, full voltage were impressed on a transformer primary with the secondary short circuited about ten times full-load current would flow. When, however, full voltage is impressed on the primary with the

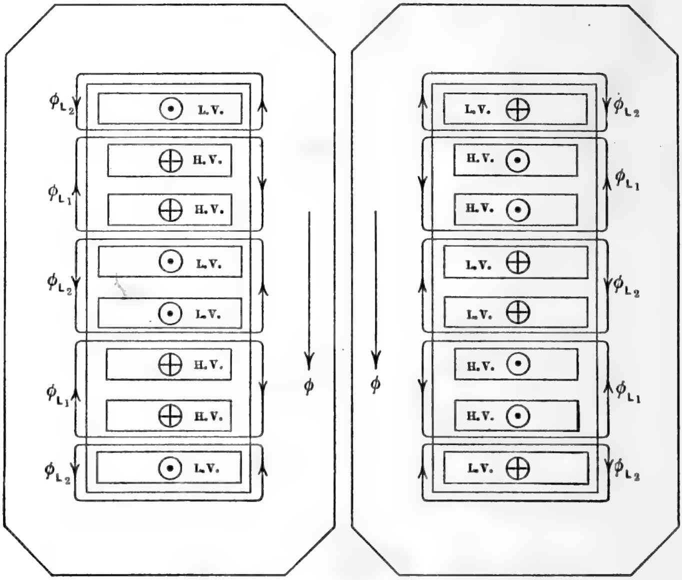


FIG. 271. Leakage fluxes in a shell-type transformer.

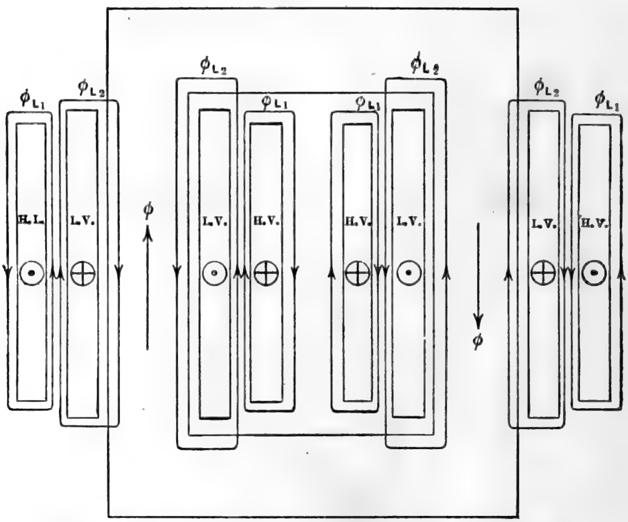


FIG. 272. Leakage fluxes in a core-type transformer.

secondary open, the exciting current which is only 10 per cent of full-load current flows. Thus the open-circuit reactance of a transformer is of the order of one hundred times the short-circuit reactance.

Fig. 273 corresponds to Fig. 267, but is drawn to different scales in order to show the relation between the various fluxes.

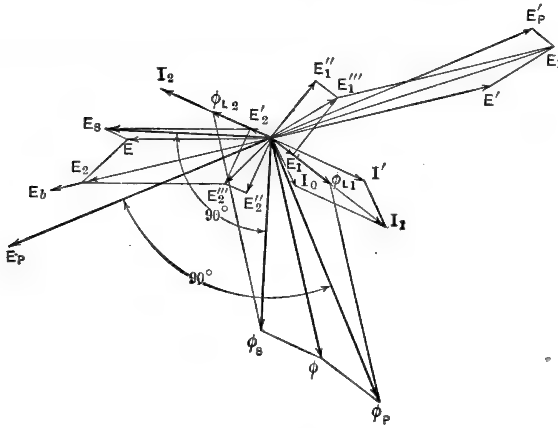


Fig. 273.

$O\Phi_{L_2} = \Phi_{L_2}$ = leakage flux surrounding the secondary winding; it induces in the secondary an e.m.f. equal and opposite to $E_2'' = jI_2x_2$.

$O\Phi_S = \Phi_S = \Phi + \Phi_{L_2}$ = total flux threading the secondary winding; it induces in the secondary an e.m.f. E_S in quadrature ahead of Φ_S.

$OE_S = E_S$ = actual e.m.f. induced in the secondary $E_S = E + E_2'$.

$O\Phi_{L1} = \Phi_{L1}$ = leakage flux surrounding the primary winding; it induces in the primary an e.m.f. equal and opposite to $E_1'' = jI_1x_1$.

$O\Phi_P = \Phi_P = \Phi + \Phi_{L_1}$ = total flux threading the primary winding; it induces in the primary an e.m.f. E_P in quadrature ahead of Φ_P.

$OE_P = E_P$ = actual e.m.f. induced in the primary.

$OE_P' = E_P'$ = component of impressed e.m.f. required to overcome E_P; $E_P' = -E_P = E_1 - E_1'$.

181. Vector Equations of the Transformer. The following equations show the relations between the various e.m.f.'s and currents in the transformer.

$$E_1 = E' + I_1 z_1 = E' + I_1 (r_1 + jx_1). \quad (280)$$

$$I_1 = I_0 + I'. \quad (281)$$

$$I_0 = E' (g_0 - jb_0). \quad (282)$$

$$I' = \frac{n_2}{n_1} I_2. \quad (283)$$

$$E' = \frac{n_1}{n_2} E_2. \quad (284)$$

Sec. ind. $$E_2 = E + I_2 z_2 = E + I_2 (r_2 + jx_2). \quad (285)$$

Sec. V. $E = I_2 z = I_2 (R + jX)$, where $Z = R + jX$ is the load impedance. (286)

$E = E (\cos \theta + j \sin \theta)$, where $\cos \theta$ is the load power factor. (287)

These equations may be combined as follows:

$$E_1 = I_1 (r_1 + jx_1) + E'.$$

$$E_1 = (I_0 + I') (r_1 + jx_1) + \frac{n_1}{n_2} E_2.$$

$$E_1 = E' (g_0 - jb_0) (r_1 + jx_1) + I' (r_1 + jx_1) + \frac{n_1}{n_2} \{E + I_2 (r_2 + jx_2)\}.$$

$$E_1 = E' (g_0 - jb_0) (r_1 + jx_1) + I' (r_1 + jx_1) + \frac{n_1}{n_2} I_2 (R + jX + r_2 + jx_2).$$

$$E_1 = E' (g_0 - jb_0) (r_1 + jx_1) + I' (r_1 + jx_1) + \left(\frac{n_1}{n_2}\right)^2 I' \{(R + r_2) + j (X + x_2)\}.$$

$$E_1 = E' (g_0 - jb_0) (r_1 + jx_1) + I' \left[(r_1 + jx_1) \cancel{(r_1 + jx)} + \left(\frac{n_1}{n_2}\right)^2 \{(R + r_2) + j (X + x_2)\} \right]. \quad (288)$$

If the exciting current is neglected, the first term may be dropped and equation 288 becomes

$$E_1 = I' \left[r_1 + jx_1 + \left(\frac{n_1}{n_2}\right)^2 \{(R + r_2) + j (X + x_2)\} \right]. \quad (289)$$

The error introduced is of the order of one per cent.

Fig. 274 represents the transformer as a circuit. If secondary quantities are represented by the equivalent primary quantities the circuit in Fig. 275 may be used to represent the transformer.

E.m.f. E_2 in the secondary = e.m.f. $E_1 = \frac{n_1}{n_2} E_2$ in the primary.

E.m.f. E in the secondary = e.m.f. $\frac{n_1}{n_2} E$ in the primary.

Current I_2 in the secondary = current $I' = \frac{n_2}{n_1} I_2$ in the primary.

Resistance r_2 in the secondary = resistance $\left(\frac{n_1}{n_2}\right)^2 r_2$ in the primary.

Resistance x_2 in the secondary = reactance $\left(\frac{n_1}{n_2}\right)^2 x_2$ in the primary.

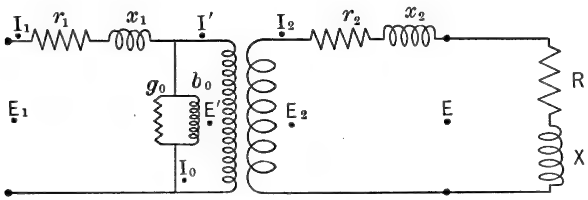


FIG. 274. Circuit diagram of a transformer.

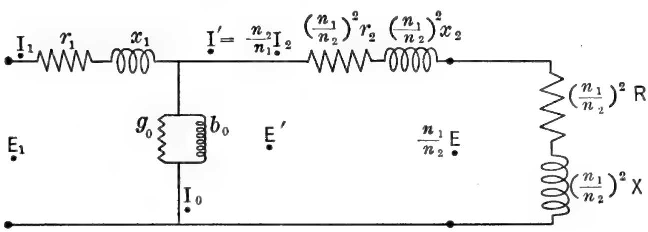


FIG. 275. Equivalent circuit diagram of a transformer.

Fig. 275 corresponds to equation 288. If the exciting current is neglected, as in equation 289, the circuit may be simplified as in Fig. 276.

From this

$$\dot{E}_1 = \dot{I}' \left\{ r_1 + jx_1 + \left(\frac{n_1}{n_2}\right)^2 (r_2 + jx_2 + R + jX) \right\},$$

or

$$\dot{E}_1 = \dot{I}' \{r + jx\} + \frac{n_1}{n_2} \dot{E},$$

where

$r = r_1 + \left(\frac{n_1}{n_2}\right)^2 r_2$ is the resistance of the transformer expressed as

an equivalent primary resistance and $x = x_1 + \left(\frac{n_1}{n_2}\right)^2 x_2$ is the reactance of the transformer expressed as an equivalent primary reactance.

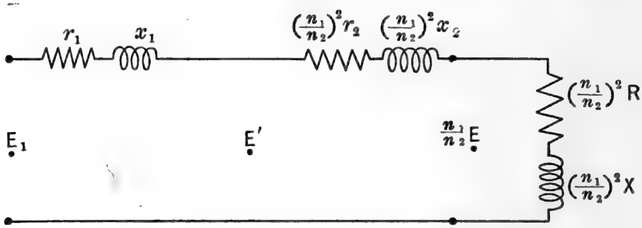


FIG. 276. Simplified circuit diagram of a transformer.

Referring to Fig. 277, which is the vector diagram of the circuit in Fig. 276,

$$E_1 = \left(\frac{n_1}{n_2} E \cos \theta + I_1 r\right) + j\left(\frac{n_1}{n_2} E \sin \theta + I_1 x\right)$$

and in absolute values

$$E_1 = \sqrt{\left(\frac{n_1}{n_2} E \cos \theta + I_1 r\right)^2 + \left(\frac{n_1}{n_2} E \sin \theta + I_1 x\right)^2}. \qquad (290)$$

By substituting for I_1 the value $\frac{n_2}{n_1} I_2$, equation 290 becomes

$$E_1 = \sqrt{\left(\frac{n_1}{n_2} E \cos \theta + \frac{n_2}{n_1} I_2 r\right)^2 + \left(\frac{n_1}{n_2} E \sin \theta + \frac{n_2}{n_1} I_2 x\right)^2}, \qquad (291)$$

which shows the relation between the secondary terminal voltage and secondary current for a given primary impressed e.m.f. and a given secondary load power factor.

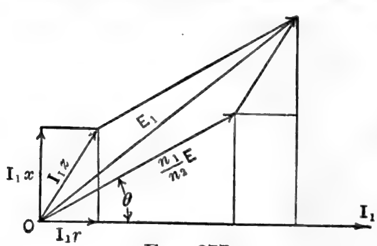


FIG. 277.

182. Examples. *I.* A step-down transformer with a ratio of turns of 10 : 1 delivers 100 kilowatts at 2000 volts to a receiver of power factor 80 per cent lagging. Determine the primary impressed voltage, the current and the power factor.

The primary impedance is

$$z_1 = r_1 + jx_1 = 50 + 80j \text{ ohms.}$$

The secondary impedance is

$$z_2 = r_2 + jx_2 = 0.6 + 0.8\,j \text{ ohms.}$$

The primary exciting admittance is

$$y_0 = g_0 - jb_0 = (2 - 6\,j)\,10^{-6}.$$

The current output or secondary current of the transformer is $\frac{1{,}000{,}000}{2000 \times 0.80} = 62.5$ and taking this as axis the various quantities may be expressed in rectangular coördinates as follows:

Secondary current

$$I_2 = 62.5 + 0j.$$

Secondary terminal e.m.f.

$E = 2000\,(\cos\theta + j\sin\theta) = 1600 + 1200\,j$, where $\cos\theta = 0.8$ is the load power factor.

Secondary impedance e.m.f.

$$E_2''' = I_2 z_2 = 62.5\,(0.6 + 0.8\,j) = 37.5 + 50\,j.$$

Secondary induced e.m.f.

$$E_2 = E + E_2''' = 1637.5 + 1250\,j.$$

Primary induced e.m.f.

$$E_b = E' = \frac{n_1}{n_2} E_2 = 10\,E_2 = 16{,}375 + 12{,}500\,j.$$

Primary exciting current

$$I_0 = E'\,y_0 = (16{,}375 + 12{,}500\,j)\,(2 - 6\,j)\,10^{-6} = 0.11 - 0.07\,j.$$

Primary load current

$$I' = \frac{n_2}{n_1} I_2 = \frac{1}{10} I_2 = 6.25 + 0\,j.$$

Total primary current

$$I_1 = I' + I_0 = 6.36 - 0.07\,j.$$

Primary impedance e.m.f.

$$E_1''' = I_1 z_1 = (6.36 - 0.07\,j)\,(50 + 80\,j) = 324 + 505\,j.$$

Primary impressed e.m.f.

$$E_1 = E' + E_1''' = 16{,}699 + 13{,}005\,j.$$

Taking absolute values,

primary impressed e.m.f.

$$E_1 = \sqrt{(16{,}699)^2 + (13{,}005)^2} = 21{,}160 \text{ volts,}$$

primary current

$$I_1 = \sqrt{(6.36)^2 + (0.07)^2} = 6.36 \text{ amperes},$$

exciting current

$$I_0 = \sqrt{(0.11)^2 + (0.07)^2} = 0.13 \text{ amperes},$$

primary induced e.m.f.

$$E' = \sqrt{(16{,}375)^2 + (12{,}500)^2} = 20{,}600 \text{ volts}.$$

The primary impressed e.m.f. is inclined to the axis of coördinates at an angle θ', where

$$\tan\theta' = \frac{13{,}005}{16{,}699} = 0.7785 \text{ and therefore } \theta' = 37°50'.$$

The primary current is inclined to the axis at an angle θ'', where

$$\tan\theta'' = -\frac{0.07}{6.36} = -\ 0.011 \text{ and therefore } \theta'' = -\ 6°\,17'.$$

The angle of phase difference between the primary current and the primary impressed e.m.f. is $\theta_1 = \theta' - \theta'' = 44°\,7'$ and the primary power factor is $\cos\theta_1 = \cos 44°\,7' = 0.722$ or 72.2 per cent.

The regulation of the transformer under these conditions of loading is $\dfrac{21{,}160 - 20{,}000}{20{,}000} \times 100$ per cent $= 5.8$ per cent.

Primary copper loss is

$$I_1^2 r_1 = (6.36)^2 \times 50 = 2020 \text{ watts}.$$

Secondary copper loss is

$$I_2^2 r_2 = (62.5)^2 \times 0.6 = 2340 \text{ watts}.$$

Iron loss is

$$E_1^2 g_0 = (20{,}600)^2 \times 2 \times 10^{-6} = 850 \text{ watts}.$$

The efficiency is therefore

$$\eta = \frac{\text{output}}{\text{output} + \text{losses}} 100 \text{ per cent}$$

$$= \frac{100{,}000}{100{,}000 + 5210} \times 100 \text{ per cent} = 95 \text{ per cent}.$$

II. If the transformer in example I, with 2000 volts impressed, is used as a step-up transformer to charge a cable system of negligible resistance and supplies 5 amperes, determine the secondary terminal e.m.f.

Primary impedance is now $\underset{\cdot}{z}_1 = 0.6 + 0.8\,j.$

Secondary impedance is $\underset{\cdot}{z}_2 = 50 + 80\,j.$

Primary exciting admittance is $\underset{\cdot}{y}_0' = 10^2\underset{\cdot}{y}_0 = (2 - 6\,j)10^{-4}.$

Let the secondary terminal e.m.f. be E and take it as the real axis, the other e.m.fs. and currents may then be expressed in rectangular coördinates.

Secondary terminal e.m.f.

$$\underset{\cdot}{E} = E + 0j.$$

Secondary current

$$\underset{\cdot}{I}_2 = 0 + 5\,j.$$

Secondary impedance e.m.f.

$$\underset{\cdot}{E}_2''' = \underset{\cdot}{I}_2\underset{\cdot}{z}_2 = 5\,j\,(50 + 80\,j) = -400 + 250\,j.$$

Secondary induced e.m.f.

$$\underset{\cdot}{E}_2 = \underset{\cdot}{E} + \underset{\cdot}{E}_2''' = E - 400 + 250\,j.$$

Primary induced e.m.f.

$$\underset{\cdot}{E}' = \tfrac{1}{10}\,\underset{\cdot}{E}_2 = 0.1\,E - 40 + 25\,j.$$

Primary load current

$$\underset{\cdot}{I}' = 10\,\underset{\cdot}{I}_2 = 0 + 50\,j.$$

Primary exciting current

$$\begin{aligned}\underset{\cdot}{I}_0' = \underset{\cdot}{E}'\underset{\cdot}{y}_0' &= (0.1\,E - 40 + 25\,j)(2 - 6\,j)10^{-4}\\ &= \{(0.2\,E + 70) - (0.6\,E - 290)\,j\}10^{-4}.\end{aligned}$$

Total primary current

$$\underset{\cdot}{I}_1 = \underset{\cdot}{I}' + \underset{\cdot}{I}_0 = [(0.2\,E + 70) - \{(0.6\,E - 290)10^{-4} - 50\}\,j].$$

Primary impedance e.m.f.

$$\begin{aligned}\underset{\cdot}{E}_1''' = \underset{\cdot}{I}_1\underset{\cdot}{z}_1 &= [(0.2\,E + 70) - \{(0.6\,E - 290)10^{-4} - 50\}\,j](0.6 + 0.8\,j)\\ &= [\{(0.6\,E - 190)\,10^{-4} - 40\} + \{(-0.2\,E + 230)\,10^{-4} + 30\}].\end{aligned}$$

Primary impressed e.m.f.

$$\underset{\cdot}{E}_1 = \underset{\cdot}{E}' + \underset{\cdot}{E}_1''' = (0.1\,E - 80) - 55\,j.$$

The absolute value of primary impressed e.m.f. is

$$E_1 = \sqrt{(0.1\,E - 80)^2 + (55)^2} = 2000$$

and solving this gives the secondary terminal e.m.f.

$$E = 23{,}160.$$

183. Measurement of the Impedance of a Transformer. The equivalent primary resistance and reactance of a transformer can most easily be obtained from a short-circuit test.

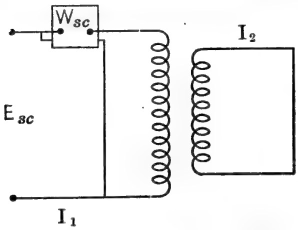


FIG. 278. Short-circuited transformer.

If E_{SC} is the impressed e.m.f. required to produce full-load current in the short-circuited secondary, Fig. 278, then E_{SC} is the impedance drop in the transformer at full load and

$$E_{SC} = I_1 z = I_1\sqrt{r^2 + x^2},$$

and the equivalent primary impedance is

$$z = \sqrt{r^2 + x^2} = \frac{E_{SC}}{I_1}.$$

The power factor may be obtained by connecting a wattmeter in the circuit; it is

$$\cos\theta_{SC} = \frac{W_{SC}}{E_{SC} I_1},$$

where W_{SC} is the power consumed in the transformer and includes the primary and secondary full-load copper losses and a very small iron loss which may be neglected.

The equivalent primary resistance is

$$r = z\cos\theta_{SC},$$

or it may be obtained as

$$r = \frac{W_{SC}}{I_1^2} = \frac{I^2 r_1 + I_2^2 r_2}{I_1^2} = r_1 + \left(\frac{n_1}{n_2}\right)^{2} r_2.$$

The equivalent primary reactance is

$$x = z\sin\theta_{SC} = x_1 + \left(\frac{n_1}{n_2}\right)^2 x_2.$$

Since the current densities in the two windings are approximately the same, the copper losses are about equal, or

$$I_1^2 r_1 = I_2^2 r_2$$

and, therefore,

$$r_1 = \left(\frac{n_1}{n_2}\right)^2 r_2 \text{ (approximately).}$$

Similarly, since the leakage paths about the two windings are similar and the m.m.fs. of the two windings are the same, the leakage fluxes are approximately equal. But the inductance or

reactance is proportional to the square of the number of turns and, therefore,

$$x_1 = \left(\frac{n_1}{n_2}\right)^2 x_2 \text{ (approximately).}$$

184. Voltage Characteristics. The voltage characteristics of a transformer show the relation between the secondary terminal

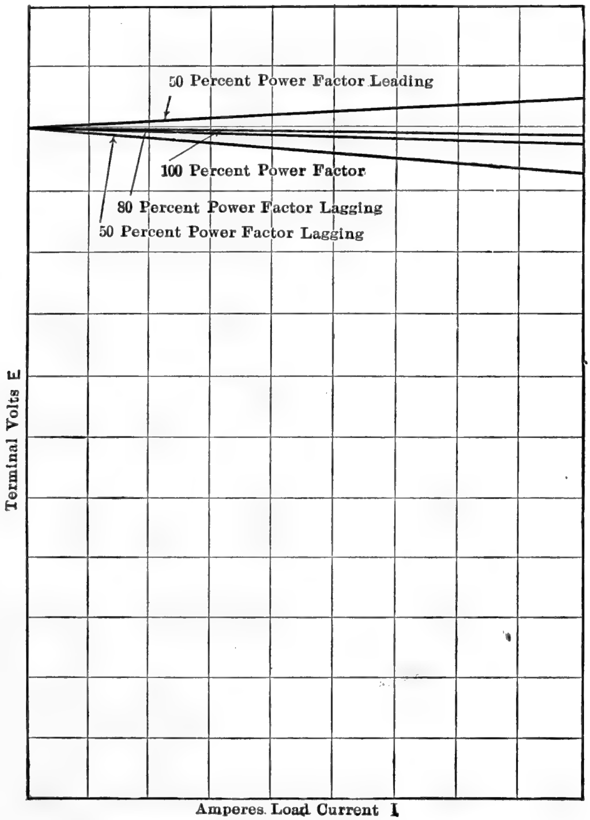


Fig. 279. Voltage characteristics of a transformer.

voltage and the secondary load current for a given primary impressed e.m.f. and a given load power factor. Typical voltage characteristics are shown in Fig. 279. They can be calculated from equation 291. The change in voltage from no load to full load is very small, even when the power factor is low.

185. Regulation. The regulation of a transformer is very much better than that of an alternator because the e.m.fs. consumed by the leakage reactances are very much smaller. If the terminal e.m.f. at full load is taken as 100 per cent and the impedance drop is expressed in per cent the sum of the two in their proper phase relation gives the terminal e.m.f. at no load in per cent.

Using rectangular coördinates and taking the secondary current as the real axis, the impedance drop in per cent is $\dot{E}_D = Ir$ per cent $+ jIx$ per cent; the secondary terminal voltage at full load is $\dot{E} = 100 \cos \theta + j\, 100 \sin \theta$; the secondary terminal voltage at no load is $\dot{E}_2 = \dot{E} + \dot{E}_D = 100 \cos \theta + Ir$ per cent $+ j\,(100 \sin \theta + Ix$ per cent),
and in absolute values

$$E_2 = \sqrt{(100 \cos \theta + Ir\%)^2 + (100 \sin \theta + Ix\%)^2} \quad . \quad (292)$$

and $\quad E = 100.$

Therefore, the per cent regulation for the load power factor $\cos \theta$ is $E_2 - E = E_2 - 100.$

Fig. 280 shows a regulation curve plotted on a power-factor base. For leading power factors the regulation becomes negative, that is, the voltage rises with load.

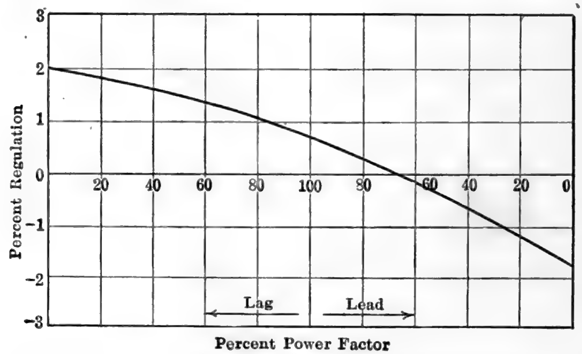


FIG. 280. Regulation curve of a transformer.

186. Losses in Transformers. The losses in a transformer are the copper losses and the iron losses.

The copper loss is

$$W_c = I_1^2 r_1 + I_2^2 r_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (293)$$

and is almost evenly divided between the primary and the secondary since the two windings are designed for approximately the same

current density. The iron loss consists of the hysteresis and eddy current losses and does not vary with load. It can be measured by applying full voltage to the primary with the secondary open and reading the watts input,

$$W_0 = E_1 I_0 \cos \theta_0 = E_1 I_c = E_1^2 g_0. \quad . \quad . \quad . \quad (294)$$

W_0 includes a small copper loss $I_0^2 r_1$ which may be neglected.

187. Hysteresis Loss. The hysteresis loss is the energy consumed in magnetizing and demagnetizing the iron. It is directly proportional to the frequency f, varies as the 1.6th power of the maximum induction density $\mathcal{B}_0$, and also depends on the quality of the iron.

The hysteresis loss per cycle per cubic centimeter of iron is

$$\omega_h = \eta \mathcal{B}_0^{1.6} \text{ ergs}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (295)$$

where η is the hysteresis constant for the iron and has a value of about 0.003 for good transformer punchings.

The hysteresis loss per second in a volume of V cu. cm. at a frequency f is

$$W_h = \eta f V \mathcal{B}_0^{1.6} \text{ ergs per second} \quad . \quad . \quad . \quad (296)$$

$$= \eta f V \mathcal{B}_0^{1.6}\, 10^{-7} \text{ watts.}$$

188. Eddy Current Loss in Transformer Iron. The eddy current loss is the energy consumed by the currents induced in the iron of the transformer by the alternating flux cutting it.

In Fig. 281

t = thickness of sheets in centimeters,
$\mathcal{B}_0$ = maximum induction density,
f = frequency,
γ = electric conductivity of iron.

AB is a section of length 1 cm., depth 1 cm. and width dx cm., parallel to the edge of the plate and distant x cm. from the centre. DC is a similar section on the other side.

The maximum flux inclosed by $ABCD$ is $2\,\mathcal{B}_0 x$ lines. Of this $\mathcal{B}_0 x$ lines cut across the section AB and generate in it an e.m.f. of effective value,

$$dE = \sqrt{2}\,\pi f \mathcal{B}_0 x \text{ c.g.s. units.}$$

The conductance of the section AB is $\gamma\, dx$, and thus the current induced in it is

$$dI = \delta E \gamma\, dx = \sqrt{2}\pi f \mathcal{B}_0 \gamma x\, dx \text{ c.g.s. units.}$$

The power consumed by the induced current in section is

$$d\omega_e = dE\,dI = 2\,\pi^2 f^2 \mathcal{B}_0{}^2 \gamma x^2\,dx \text{ ergs per second}$$

and thus the energy consumed per square centimeter of the plate is

$$\begin{aligned}\omega_e &= 2\,\pi^2 f^2 \mathcal{B}_0{}^2 \gamma \int_{-\frac{t}{2}}^{\frac{t}{2}} x^2\,dx \\ &= 2\,\pi^2 f^2 \mathcal{B}_0{}^2 \gamma \left[\frac{x^3}{3}\right]_{-\frac{t}{2}}^{\frac{t}{2}} \\ &= \pi^2 f^2 \mathcal{B}_0{}^2 \gamma \frac{t^3}{6} \text{ ergs per second.} \quad \quad (297)\end{aligned}$$

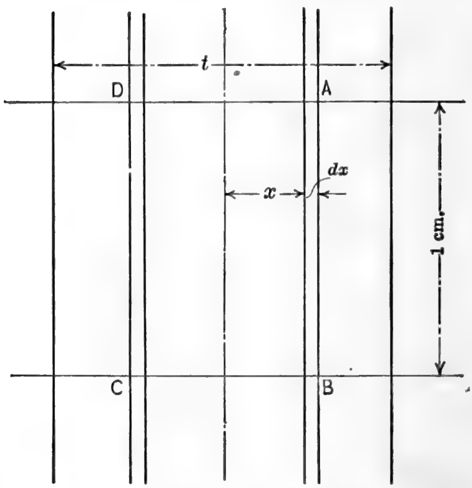


FIG. 281. Eddy currents in sheet iron.

The power consumed per cubic centimeter of iron is

$$\omega_1 = \frac{\omega_e}{t} = \frac{\pi^2 f^2 \mathcal{B}_0{}^2 \gamma t^2}{6} \text{ ergs per second.} \quad \quad (298)$$

Thus the eddy current loss for a volume of V cubic centimeter is

$$\begin{aligned}W_e &= \frac{\pi^2 f^2 \mathcal{B}_0{}^2 \gamma t^2 V}{6} \text{ ergs per second} \\ &= \frac{\pi^2 f^2 \mathcal{B}_0{}^2 \gamma t^2 V}{6} 10^{-7} \text{ watts.} \quad \quad (299)\end{aligned}$$

γ has a value of about 10^5 for sheet iron.

The eddy current loss, therefore, varies as the square of the frequency, the square of the maximum induction density and the square of the thickness of the plates and it also depends on the electric conductivity of the iron.

To reduce the eddy current loss the core is built up of sheets 0.014 in. thick and an iron with a high electric resistance is used.

Since iron has a positive temperature coefficient for resistance, the eddy current loss will decrease slightly as the transformer heats up.

189. Efficiency. The efficiency of transformers is very high. For small sizes it varies from 93 per cent at $\frac{1}{4}$ load to 97 per cent at full load and for large sizes from 97 per cent at $\frac{1}{4}$ load to 99 per cent at full load.

Fig. 282 shows the loss and efficiency curves for a large transformer.

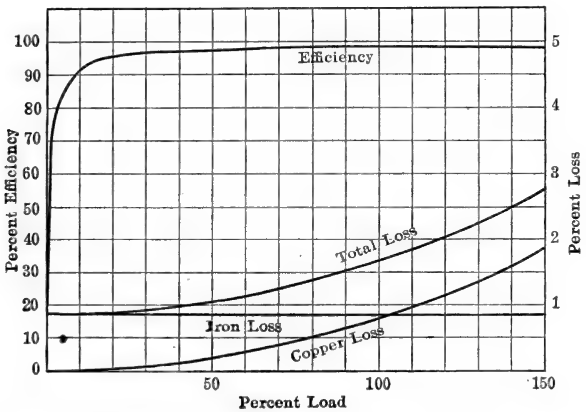


Fig. 282. Efficiency and loss curves of a 550-kw., 10,500-volt, 60-cycle, air-blast transformer.

190. Types of Transformers. Transformers may be divided into two general types according to the arrangement of the core and windings, the core type and the shell type. These two types are illustrated in Fig. 283 and Fig. 284.

Three-phase transformers are built of both the core and shell types, Fig. 285 and Fig. 286. In the great majority of polyphase systems, however, groups of single-phase transformers instead of three-phase units are used. This gives a more flexible system and there is less danger of shut down, since an injury to a single-phase

transformer in a group does not necessarily shut down the line fed by that group, but operation can be carried on more or less satisfactorily with two transformers in open delta. An injury to one coil of a three-phase transformer puts the whole transformer out of commission. Thus for the same reliability of operation the cost of spare apparatus is increased with three-phase units.

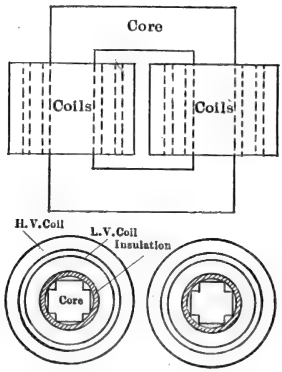


FIG. 283. Core-type transformer.

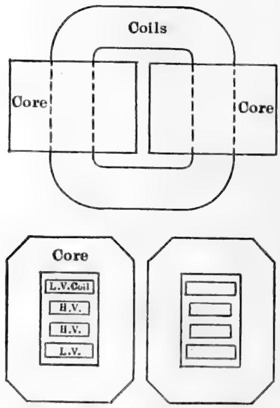


FIG. 284. Shell-type transformer.

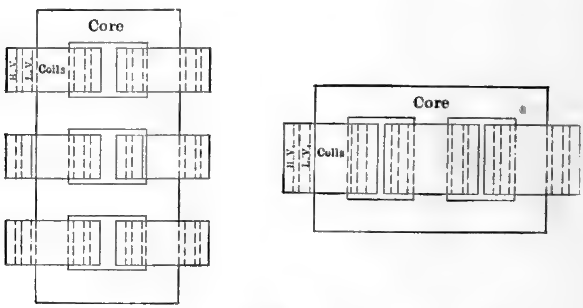


FIG. 285. Three-phase core-type transformers.

191. Methods of Cooling. Very small transformers do not require any special method of cooling but are so designed that the exposed surface is large enough to radiate the heat generated by the power losses in the windings and core without a temperature rise exceeding the limits consistent with the life of the insulation. Transformers must be designed to operate for 24 hours at

full load with a temperature rise not exceeding 40° C. above standard room temperature of 25° C.

Since the output and losses in a transformer increase in proportion to its volume or as the cube of its linear dimensions while the radiating surface increases only as the square of the linear dimensions, as the output is increased special methods of cooling must be adopted.

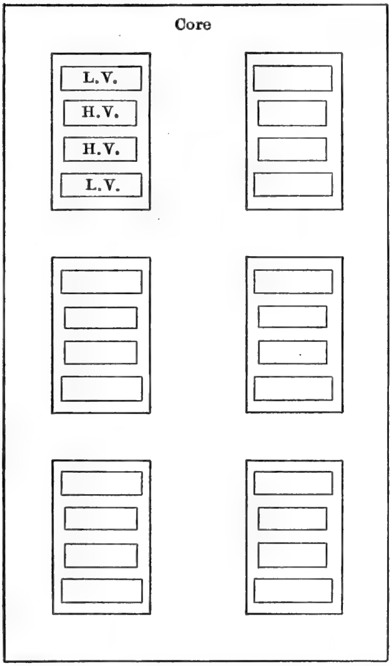


FIG. 286. Three-phase shell-type transformer.

Transformers up to 300 kw. are usually immersed in tanks containing oil of good insulating qualities. This oil serves the double purpose of increasing the insulation of the transformer and conducting away the heat developed by the losses. Such transformers are called oil-insulated self-cooled transformers. The cases are made with deep corrugations to give a larger radiating surface exposed to the air.

A second method of cooling is to pump air through the transformer to carry off the heat developed. Transformers cooled in this way are termed "air blast" and are used on electric locomotives where their light weight is an advantage and in places where oil cannot be used on account of the danger of fire. They cannot be operated satisfactorily above 30,000 volts as the insulation rapidly deteriorates due to ozone set free in air at high voltages. Oil is then a necessary protection.

For air-blast transformers about 150 cu. ft. of air per minute are required for each kilowatt lost, to keep the temperature within the 40° C. rise allowed.

Large transformers, above 300 kw., are cooled by placing in the upper part of the tank cooling coils which carry a continuous flow of cold water, which conveys the heat away from the oil.

With water at 25° C. about $\frac{1}{4}$ gallon per minute per kilowatt lost is required for 40° C. rise. If too much water is used the transformer will be cooled below the temperature of the air and moisture may collect on it and cause a breakdown of the insulation. The cooling coils should have a surface in contact with the oil of about 0.9 sq. in. per watt lost.

192. Transformer Connections. If the primary and secondary windings are each divided into a number of coils, which

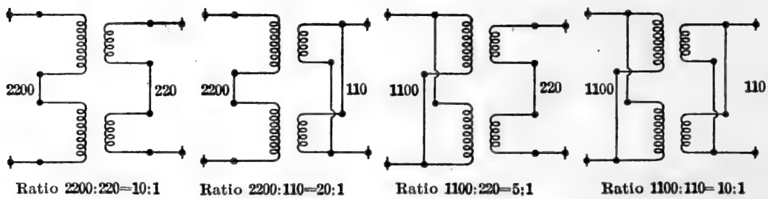


FIG. 287. Transformer connections.

can be connected either in series or parallel, a number of different ratios of transformation can be obtained.

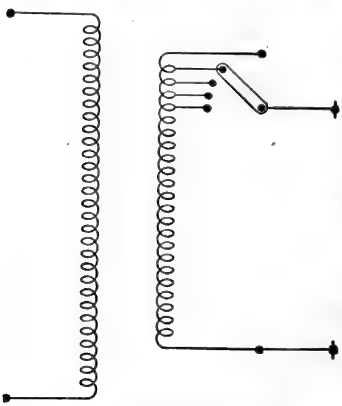


FIG. 288. Secondary with taps.

Take for example a standard lighting transformer with two coils for 110 volts on the low-voltage side and two coils for 1100 volts on the high-voltage side. The four possible connections are shown in Fig. 287.

If small percentage changes of ratio are required for line regulation a number of taps are brought out from the primary or secondary winding so that the number of turns in use may be changed by the required amount. (Fig. 288.)

193. Single-phase Transformers on Polyphase Circuits. Single-phase transformers are used in groups on polyphase circuits.

On three-phase circuits the transformers may be grouped in four ways as shown in Fig. 289. Both the primaries and secondaries may be connected in "Y" or "Δ".

The ΔΔ arrangement is used where the voltages are not extremely high and where a neutral point is not necessary. It

has the advantage over the other connections that, if one transformer burns out the system can still be operated with the two remaining transformers connected in V but carrying a considerable increase of load. The V connection is discussed in Art. 194 below.

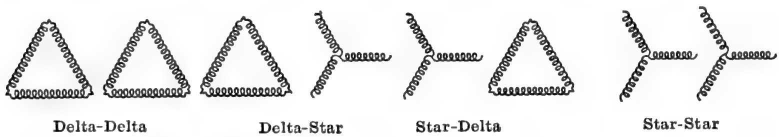


FIG. 289. Single-phase transformers on three-phase circuits.

The ΔY arrangement is used at the generating end of high-voltage transmission systems so that the voltage per transformer may be as low as possible. The system has a neutral point on the high-voltage side which may be grounded. The $Y\Delta$ arrangement is used at the receiver end of the line.

The YY arrangement may be used in place of any of the above and gives a neutral point on both the high-voltage and low-voltage sides.

194. "Open Delta" or "V" Connection. If one transformer of a $\Delta\Delta$ system burns out, the system can still be operated

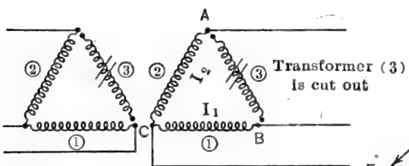


FIG. 290. Open delta or "V" connection.

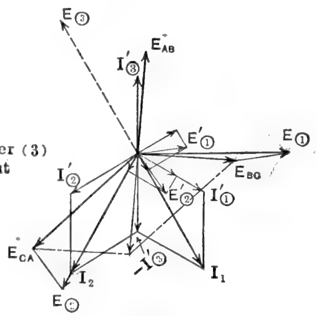


FIG. 291.

with the other two transformers without any change in connections. This arrangement of two transformers on a three-phase system, Fig. 290, is called the "Open Delta" or "V"-connection. Each transformer carries the full line current and therefore for the same output from the system the current in the transformer windings is increased in the ratio $\sqrt{3} : 1$ or 73 per cent.

To supply a load of 300 kv.a. from a three-phase system three 100 kv.a. transformers are required if connected in "delta" or two 173 kv.a. transformers if connected in "open delta."

In the vector diagram in Fig. 291 E_1, E_2 and E_3 are the e.m.f.'s generated in the three transformers of a "delta" system; they are also equal to the terminal e.m.f.'s at no load in the "open delta." I_1', I_2', I_3' are the load currents with a closed delta. Transformer (3) is disconnected and the two remaining transformers (1) and (2) carry currents I_1 and I_2 respectively and the voltages between the lines are E_{BC}, E_{CA} and E_{AB}. These voltages are never exactly balanced and they become more and more unbalanced as the load is increased.

195. Transformation from Two Phase to Three Phase. The "Scott" connection or "T" connection is an arrangement of two transformers by means of which three-phase power can be obtained from a two-phase system or the reverse. (Fig. 292.)

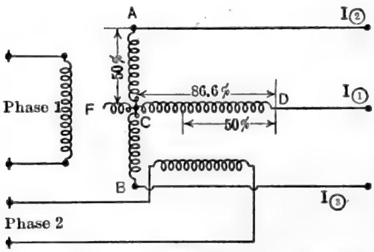


FIG. 292. Two-phase to three-phase.

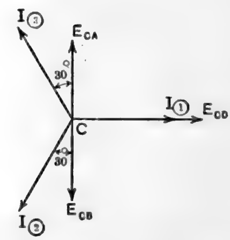


FIG. 293. Non-inductive load.

Two similar transformers are used with their primaries connected to the two-phase system. The secondary AB of one is tapped at its middle point and the secondary FD of the other has a tap at a point 86.6 per cent from the end D. The two taps are connected at C. The part FC of the winding of the second transformer is not used.

Three-phase power can be taken off at the terminals A, B and C.

If the e.m.fs. generated in the winding FD is

$$e_{FD} = E_0 \sin \theta,$$

then the e.m.fs. generated in the various sections are

$$e_{CD} = 0.866\, E_0 \sin \theta,$$
$$e_{CA} = 0.5\, E_0 \sin (\theta - 90),$$
$$e_{CB} = 0.5\, E_0 \sin (\theta + 90),$$

and, therefore,

$$e_{DA} = e_{CD} - e_{CA} = \frac{\sqrt{3}}{2} E_0 \sin\theta + \frac{1}{2} E_0 \cos\theta$$
$$= E_0 \sin(\theta + 30),$$
$$e_{AB} = E_0 \sin(\theta - 90),$$
$$e_{BD} = e_{CB} - e_{CD} = \frac{1}{2} E_0 \cos\theta - \frac{\sqrt{3}}{2} E_0 \sin\theta$$
$$= E_0 \sin(30 - \theta) = E_0 \sin\theta + 150.$$

Therefore the three e.m.fs. E_{DA}, E_{AB} and E_{BD} are displaced 120 degrees in phase and are equal and thus form a three-phase system.

Under load the e.m.fs. will not be exactly balanced since even at non-inductive load the current in one-half of the winding AB will lead the e.m.f. and in the other half will lag behind it as shown in Fig. 293.

The power delivered is

$$P = CD \cdot I_1 \cos\theta + CB \cdot I_2 \cos(30 + \theta) + CA \cdot I_3 \cos(30 - \theta)$$
$$= \frac{\sqrt{3}}{2} EI \cos\theta + \frac{1}{2} EI \cos(30 + \theta) + \frac{1}{2} EI \cos(30 - \theta)$$
$$= \frac{\sqrt{3}}{2} EI \cos\theta + \frac{\sqrt{3}}{2} EI \cos\theta = \sqrt{3}\, EI \cos\theta,$$

and one half of the power is delivered by each transformer. The neutral point of the system is at O the centre of the winding FD.

196. Series Transformer. Series transformers are used to insulate ammeters, wattmeters, relays, etc., from high-voltage circuits, or to reduce the line current to a value suitable for such instruments. In all of these instruments it is very important that the ratio of the currents in the two windings should remain constant throughout the full range. In the case of the wattmeter it is also necessary that the primary and secondary currents should be exactly in line, that is, that the phase shift of the transformer should be as small as possible.

Fig. 294 and Fig. 295 show vector diagrams for series transformers with the secondaries closed through the meter. The resistance and inductance of the meter are included in the impedance of the transformer. In Fig. 294 the secondary reactance has been taken as double the resistance and in Fig. 295 the resistance is double

the reactance. The primary and secondary currents are the same in both cases and the secondary impedance is the same.

The only voltage generated in the secondary is the impedance drop and therefore the flux and the exciting current are the same in both cases. The angle θ_0 measures the phase shift of the transformer; it is the amount by which the primary and secondary currents are out of line. Referring to the diagrams; with the large value of reactance the phase shift is small but the ratio of currents tends to differ from the ratio of turns; with the large resistance

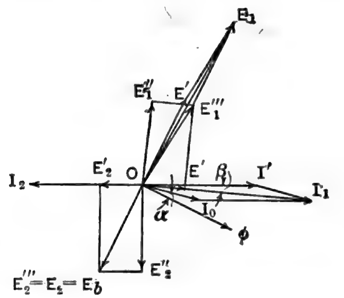


FIG. 294. Vector diagram of a current transformer with large secondary reactance.

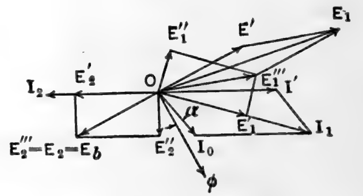


FIG. 295. Vector diagram of a current transformer with a large secondary resistance.

the phase shift is increased but the ratio of currents is more nearly constant and equal to the ratio of turns. In order to fulfil both the required conditions the exciting current and the core loss should be as small as possible and therefore the core should have high permeability. In order to reduce the flux to a minimum the impedance of the secondary should be as small as possible. The impedance of the primary winding does not affect the instrument except in so far as the primary leakage reactance is a measure of the secondary leakage reactance.

The primary winding usually consists of a single turn in series with the line and the secondary may have any number of turns depending on the reduction in current required.

The current in the primary is the load current of the circuit and does not depend in any way on the transformer. It sets up a magnetic flux in the core which links with both the primary and secondary windings. When the secondary is open circuited all the primary m.m.f. is magnetizing and a very large flux is produced in the core. This may produce dangerous voltages in the

secondary, often reaching thousands of volts in transformers with a large ratio of turns.

When the secondary is closed a current flows in it and produces a m.m.f. opposing the primary m.m.f. and reducing the flux to a very small value.

The primary e.m.f. is the drop of voltage in the transformer due to its impedance and is very small, since the secondary is short circuited.

The iron core is only designed to carry the flux required with a closed secondary and it would be very highly saturated under open-circuit conditions and would become hot due to the excessive iron losses.

197. Auto-Transformer. An auto-transformer has only one winding; the primary includes all the turns while the secondary includes only a part of them. The secondary voltage is usually made variable by bringing out a number of tops.

Auto-transformers are used very extensively to obtain a variable voltage for starting induction motors, synchronous motors, single-phase series motors, etc., and as balance coils on three-wire distributing circuits.

Fig. 296 shows an auto-transformer for starting and operating a single-phase series motor from a high-voltage trolley in electric railway service.

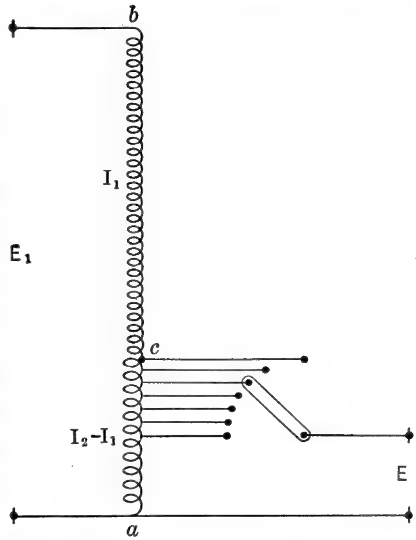


FIG. 296. Auto-transformer for single-phase electric railway motors.

The transformer in Fig. 297 is suitable as an induction motor starter. For three-phase motors three auto-transformers connected "star" may be used or two connected in "open delta." The unbalance of voltage in the open delta is not objectionable, since the transformers are only employed for starting.

Fig. 298 shows an auto-transformer used as a balance coil on a 220-volt, three-wire system.

In Fig. 296 the part of the winding from b to c carries only the primary current, while the part from a to c carries the difference between the secondary and primary currents. When the ratio of

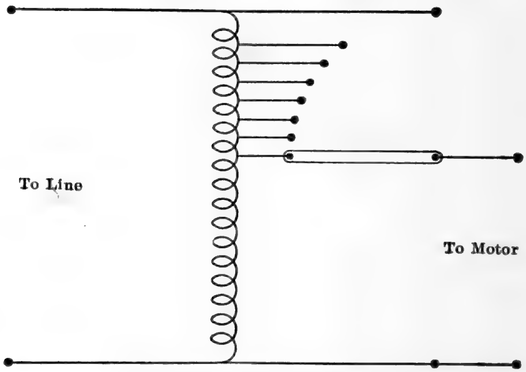


FIG. 297. Auto-transformer as induction motor starter.

turns is high the current from a to c is much larger than the primary current and this part of the winding must be made of large enough section to carry it.

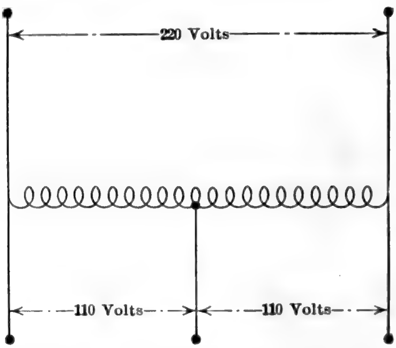


FIG. 298. Auto-transformer as a balance coil for a three-wire system.

The copper loss in an auto-transformer is smaller than in an equivalent two-coil transformer and the efficiency is therefore higher but this advantage decreases as the ratio of turns increases. With a high ratio of turns it is a disadvantage not to be able to insulate the low-voltage side of the transformer from the high-voltage side.

An auto-transformer may be used as a step-up transformer by connecting the low-voltage side to the supply.

198. The Constant-current Transformer. The constant-current transformer is shown diagrammatically in Fig. 299. The primary coil P is fixed in position and receives power at constant voltage. The secondary coil S is movable and regulates for constant current in the receiver circuit which it supplies irrespec-

tive of the load. The transformer is used to obtain a constant current for series arc-light circuits.

When the secondary coil is close to the primary there is very

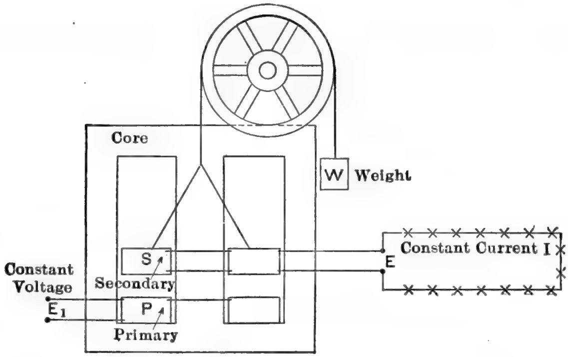


FIG. 299. Constant-current transformer.

little leakage and most of the flux produced by the primary links with the secondary and the secondary voltage is, therefore, a maximum. Primary and secondary currents are in opposite

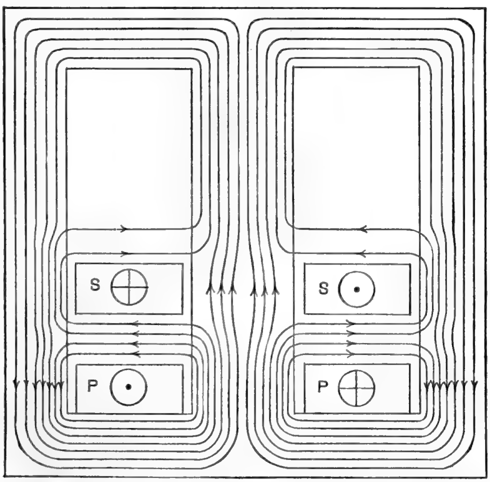


FIG. 300. Flux in the core of a constant-current transformer.

directions and the two coils repel one another. The weight W is so adjusted that the pull due to it together with the force of repulsion of the coils just balances the weight of S and allows

the coil to remain in such a position that the required current flows in it. Fig. 300 shows the flux in the core.

If the resistance or impedance of the load circuit decreases due to the cutting out of one or more arc lamps an increase of the current in both secondary and primary follows and the repulsion between the coils, which is proportional to the product of their currents, increases. The secondary, therefore, rises and increases the leakage reactances of both coils and so less of the primary magnetism links with the secondary; its voltage is, therefore, decreased and its current drops to the required value. The moving arm must be designed to give the required regulation with a fixed weight W.

Such an arrangement regulates for constant current between the limits of secondary voltage set by the two extreme positions of the moving coil.

Neglecting the primary exciting current, equation 289, Art. 181, applies to the constant-current transformer.

$$\dot{E}_1 = \dot{I}' \left\{ r_1 + jx_1 + \left(\frac{n_1}{n_2}\right)^2 (r_2 + jx_2) \right\} + \frac{n_1}{n_2}\dot{E}$$

$$= \dot{I}_2 \left\{ \frac{n_2}{n_1}(r_1 + jx_1) + \frac{n_1}{n_2}(r_2 + jx_2) \right\} + \frac{n_1}{n_2}\dot{E}.$$

$\dot{E}_1$, I_2, r_1 and r_2 are constant and x_1 and x_2 increase as E decreases.

199. Induction Regulator. Induction regulators are special transformers used to vary the voltage of an alternating-current distributing circuit or the voltage impressed on a rotary converter.

There are two types of induction regulators, single-phase and polyphase.

(1) The single-phase regulator is illustrated in Figs. 301 to 303. The primary coil P is carried on a movable core built of laminated iron and is connected across the line. The secondary coil S is carried on a stationary core and is connected in series with the line to raise or lower the voltage. (Fig. 304.)

The primary exciting current produces an alternating flux, which induces a voltage in the secondary. This secondary voltage varies with the position of the primary winding, but it is always in phase with or in opposition to the impressed voltage or line voltage. In Fig. 301 all the primary flux (neglecting leakage) passes through the secondary coil and the secondary voltage is a

maximum and is in opposition to the impressed voltage, and so gives the minimum load voltage.

In Fig. 305 OP is the impressed voltage or line voltage, OS_1 is the maximum secondary induced voltage and S_1P is the load voltage.

The load current of the circuit flows in the secondary, and there must also be a load current in the primary of equal and opposite m.m.f. as in the ordinary transformer.

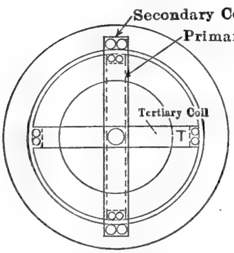


FIG. 301.

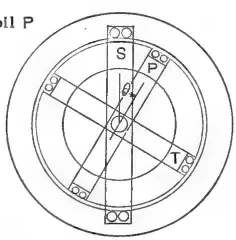


FIG. 302.

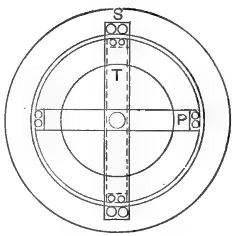


FIG. 303.

Single-phase induction regulator.

When the movable core is turned through an angle θ, Fig. 302, only part of the primary flux passes through the secondary and the secondary voltage is reduced approximately in the ratio $1 : \cos \theta$, but is still in opposition to the line voltage. The load voltage is represented by S_2P, Fig. 305.

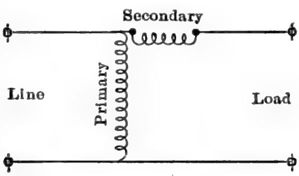


FIG. 304.

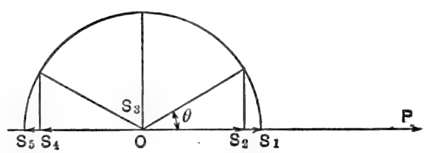


FIG. 305.

With the primary coil at right angles to the secondary coil, Fig. 303, none of the primary flux passes through S and there is no secondary induced voltage.

When the core is turned through 180 degrees the secondary voltage is again maximum, but is in phase with the impressed voltage and so raises the load voltage to S_5P. Thus the total variation of the load voltage is from S_1P to S_4P and is equal to twice the maximum secondary voltage.

In Fig. 303 the m.m.f. of the load current in the secondary cannot be opposed by any primary m.m.f. since the coils are at right angles. To supply the m.m.f. required to balance the secondary load m.m.f., and so prevent a large reactance drop in the winding, the coil T called the "tertiary" coil is placed on the movable core at right angles to the primary coil. It is short circuited and exerts an m.m.f. equal and opposite to the secondary m.m.f., and so reduces the secondary reactance to the value corresponding to the leakage flux.

The only current carried by the primary coil in this position is the exciting current.

In intermediate positions of the rotor the secondary m.m.f. is partly balanced by the induced current in the tertiary or compensating coil and partly by a load current in the primary coil.

(2) The polyphase induction regulator has a polyphase winding on the moving core, which is connected to the polyphase supply. The secondary or stator is wound with the same number of phases, but the phase windings are kept entirely separate so that they can be connected in the different lines to raise or lower the voltage.

When polyphase currents flow in the primary windings a revolving magnetic field is produced of constant value as in the alternator or induction motor. This field cuts the secondary windings and generates e.m.fs. in them of the same frequency as the primary impressed e.m.fs., but less in the ratio of turns.

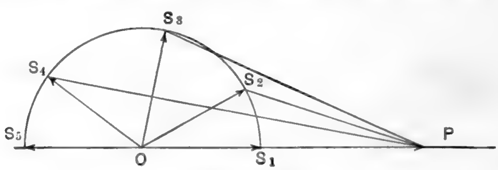


FIG. 306.

As the primary is turned the magnitude of the revolving field is not changed, and therefore the magnitude of the secondary e.m.fs. is not changed but their phase relations with the impressed e.m.fs. are changed and the load voltage is varied as shown in Fig. 306. By turning the primary through 180 degrees the phase of the secondary e.m.f. is changed from direct opposition to the line voltage to direct addition to it and thus the load voltage is varied by an amount equal to twice the secondary voltage.

The rotor must be clamped in the required position or it will tend to rotate at a high speed in the direction opposite to that of the revolving field.

The advantage of the induction regulator over a transformer with variable voltage taps on the secondary or primary is that the variation of voltage is uniform over the entire range. Regulators are, however, very expensive and require a large exciting current and have large leakage reactances.

The regulator is operated either by hand or by means of a small motor placed on the top or it can be made automatic if required.

CHAPTER. VII

INDUCTION MOTOR

200. Induction Motor. Fig. 307 shows an induction motor of the squirrel-cage type. It consists of two main parts, the primary or stator and the secondary or rotor and is a combination of a synchronous motor and a transformer. The stator is exactly similar to the armature of a synchronous motor. The

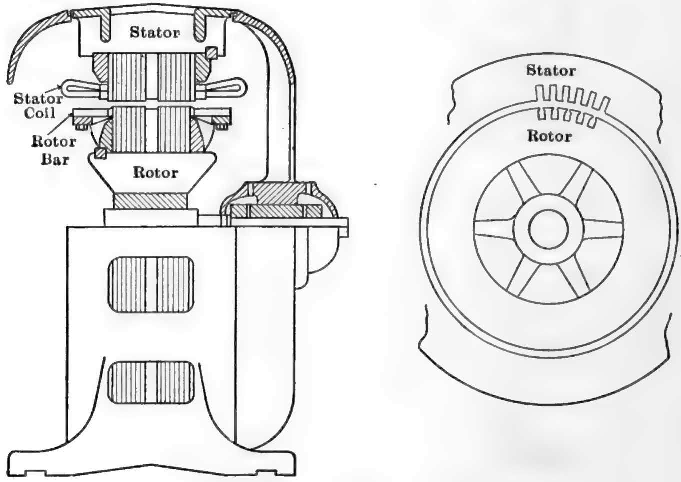


FIG. 307. Induction motor.

rotor which takes the place of the rotating field member of the synchronous motor is not excited by direct current but has currents induced in it by transformer action from the stator; thus the transfer of power from the stator to the rotor is similar to the transfer of power from the primary to the secondary in a transformer. The rotor is however free to move and there is an air gap in the magnetic circuit and therefore the magnetizing current is large, the leakage reactances are large and the power factor is low.

201. The Stator. The primary or stator consists of a winding carried in slots on the inner face of a laminated iron core. The winding is similar to an alternator or synchronous motor winding and the coils are connected in groups according to the number of phases and poles, one group per phase per pair of poles.

The stator is supplied with polyphase alternating currents and a revolving m.m.f. is produced similar to the m.m.f. of armature reaction in an alternator, which produces a magnetic field revolving at a constant speed called the synchronous speed of the motor.

In Fig. 308 (*a*) represents the stator winding of a two-pole,

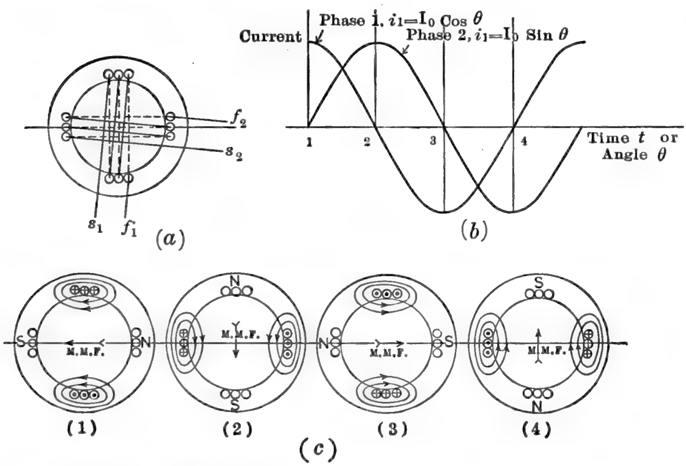


FIG. 308. Revolving m.m.f. and flux in a two-pole, two-phase induction motor.

two-phase induction motor, (*b*) the currents supplied to the two phases and (*c*) the fluxes produced by the resultant stator m.m.f. at the points (1), (2), (3) and (4) of the cycle.

The windings start at s_1 and s_2 and finish at f_1 and f_2, respectively. A positive current is one which enters at s_1 or s_2 and a negative current is one which enters at f_1 or f_2.

Referring to Fig. 308(*c*) it is seen that the north pole makes one complete revolution in the anti-clockwise direction while the current in phase 1 passed through one cycle.

Fig. 309(*a*) represents the stator of a two-pole three-phase

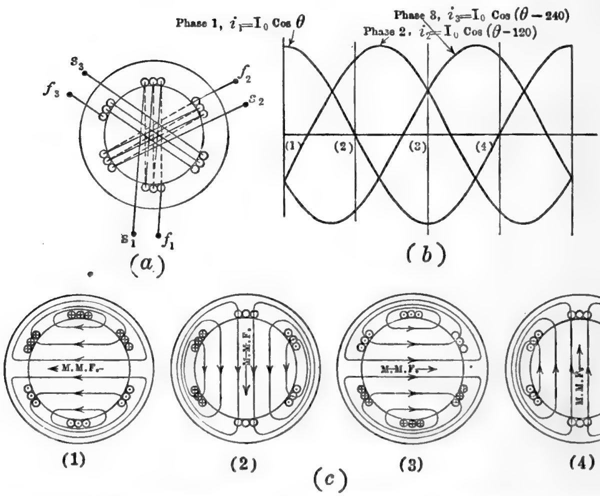


FIG. 309. Revolving m.m.f. and flux in a two-pole, three-phase induction motor.

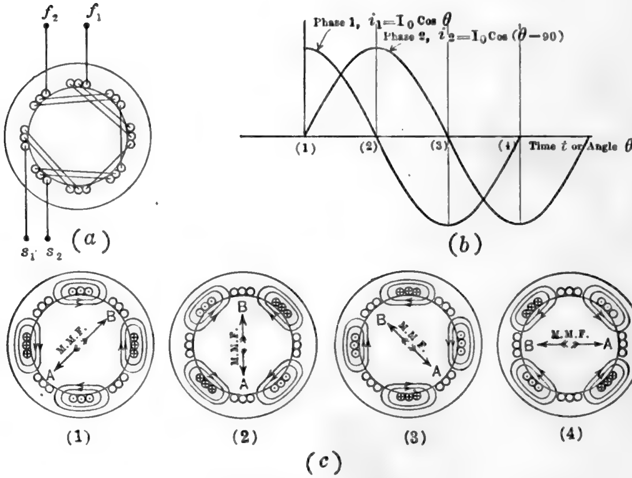


FIG. 310. Revolving m.m.f. and flux in a four-pole, two-phase induction motor.

induction motor, 309(*b*) the currents supplied and 309(*c*) the fluxes corresponding to the points (1), (2), (3) and (4) on the cycle. The north pole as before makes one revolution during one cycle.

Fig. 310(*a*) represents the stator of a four-pole, two-phase induction motor, (*b*) the currents supplied and (*c*) the fluxes produced.

Fig. 311(*a*), (*b*) and (*c*) represent a similar set of conditions for a four-pole, three-phase stator.

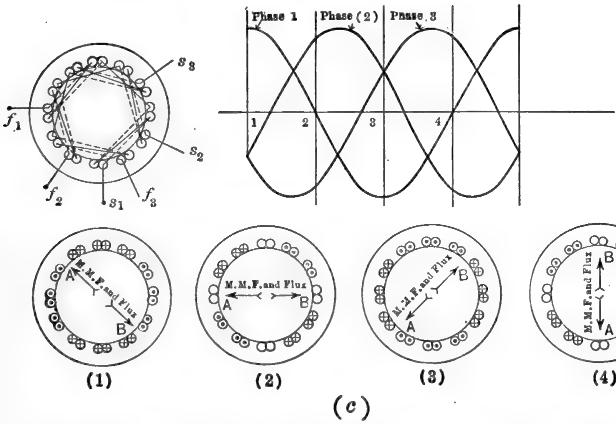


FIG. 311. Revolving m.m.f. and flux in a four-pole, three-phase induction motor.

In Fig. 310(*c*) and Fig. 311(*c*) it is seen that, while the current goes through one cycle, the revolving field turns through the angle occupied by one pair of poles.

If the stator winding has p-poles, the revolving field turns through $\frac{360}{p/2}$ degrees, that is, through $\frac{2}{p}$ of one revolution during one cycle of the current.

If the frequency of the supply is f cycles per second the revolving field makes $\frac{2f}{p}$ r.p.s. or $\frac{120f}{p}$ r.p.m. The synchronous speed of an induction motor is, therefore,

$$N = \frac{120f}{p} \text{ r.p.m.} \quad \ldots \ldots \quad (300)$$

202. Revolving M. M. F. and Flux of the Stator. In Fig. 312 OX is the direction of the m.m.f. of phase 1 of the two-phase motor in Fig. 308 and its value at any instant is

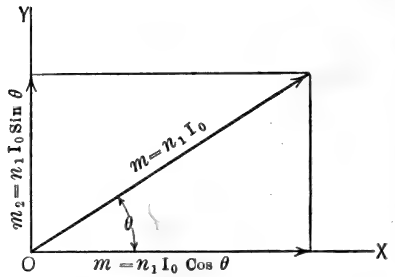


FIG. 312.

$$m_1 = n_1 I_0 \sin(\theta + 90) = n_1 I_0 \cos\theta,$$

where n_1 is the number of turns per phase and $i_1 = I_0 \sin(\theta + 90)$ is the current in phase 1.

At the same instant the m.m.f. of phase 2 is

$$m_2 = n_1 I_0 \sin\theta \text{ in direction } OY,$$

where $$i_2 = I_0 \sin\theta \text{ is the current in phase 2.}$$

The resultant m.m.f. of the two phases is

$$m = \sqrt{m_1^2 + m_2^2} = n_1 I_0 \sqrt{\cos^2\theta + \sin^2\theta} = n_1 I_0$$

and makes an angle θ with the OX axis.

The resultant m.m.f. is, therefore, constant in value, being equal to the maximum m.m.f. of one phase, and it revolves at synchronous speed in the anti-clockwise direction.

This constant m.m.f. acting on a path of constant reluctance produces a field of constant strength revolving with the m.m.f. and, therefore, revolving at synchronous speed relative to the winding of the stator. The flux linking with each phase of the stator is an alternating flux which reaches its maximum when the current in the phase is maximum and is therefore in phase with it.

Figs. 309(a), (b) and (c) represent respectively the winding of a three-phase, two-pole stator, the currents supplied and the m.m.fs. produced. Fig. 313 shows the m.m.fs. of the three phases as vectors.

The currents are

$$i_1 = I_0 \cos\theta, \text{ in phase 1,}$$

$$i_2 = I_0 \cos(\theta - 120), \text{ in phase 2,}$$

$$i_3 = I_0 \cos(\theta - 240), \text{ in phase 3.}$$

The m.m.f. of phase 1 is $n_1I_0 \cos\theta$ in direction OA.
The m.m.f. of phase 2 is $n_1I_0 \cos(\theta - 120)$ in direction OB.
The m.m.f. of phase 3 is $n_1I_0 \cos(\theta - 240)$ in direction OC.
The resultant m.m.f. in the horizontal direction is

$$m_X = n_1I_0\cos\theta + n_1I_0\cos(\theta - 120)\cos 120 + n_1I_0\cos(\theta - 240)\cos 240$$

$$= n_1I_0\left\{\cos\theta - \frac{1}{2}(\cos\theta\cos 120 + \sin\theta\sin 120) - \frac{1}{2}(\cos\theta\cos 240 + \sin\theta\sin 240)\right\}$$

$$= n_1I_0\frac{3}{2}\cos\theta = \frac{3}{2}n_1I_0\cos\theta.$$

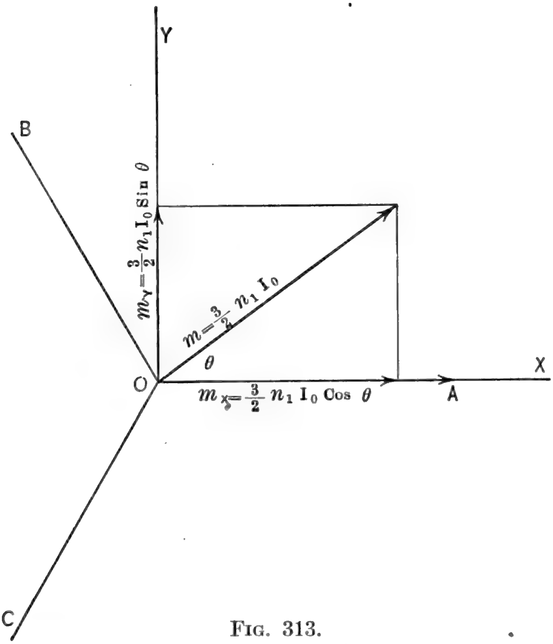


FIG. 313.

The resultant m.m.f. in the vertical direction is

$$m_Y = n_1I_0\{\cos(\theta - 120)\sin 120 + \cos(\theta - 240)\sin 240\}$$

$$= n_1I_0\left\{\frac{\sqrt{3}}{2}(\cos\theta\cos 120 + \sin\theta\sin 120) - \frac{\sqrt{3}}{2}(\cos\theta\cos 240 + \sin\theta\sin 240)\right\}$$

$$= n_1I_0\frac{\sqrt{3}}{2}\sqrt{3}\sin\theta = \frac{3}{2}n_1I_0\sin\theta.$$

The resultant m.m.f. of the three phases is

$$m = \sqrt{m_X^2 + m_Y^2} = \tfrac{3}{2}\, n_1 I_0 \sqrt{\cos^2\theta + \sin^2\theta} = \tfrac{3}{2}\, n_1 I_0$$

and makes an angle θ with the OX axis.

The resultant m.m.f. is, therefore, constant in value being equal to $\frac{3}{2}$ times the maximum m.m.f. of one phase and it revolves at synchronous speed.

This constant m.m.f. produces a field of constant strength revolving at synchronous speed. The revolving field links successively with the windings and generates e.m.fs. in them. The flux linking with any phase is maximum when the current in that phase is maximum and, therefore, the flux and current are in phase.

If a four-pole stator, Fig. 310, had been chosen instead of the two-pole stator, the m.m.fs. of the two-phase windings would have been combined at 45 degrees instead of 90 degrees and the resultant m.m.f. and flux would not remain constant but would pulsate four times during each revolution. The flux threading any phase would, however, still vary according to a sine law and would be in phase with the current in that phase.

To reverse the direction of rotation of a two-phase induction motor, it is necessary to reverse one phase only.

To reverse a three-phase motor any two leads may be interchanged.

203. The Rotor. The secondary or rotor is made in two forms, (a) the wound rotor and (b) the squirrel-cage rotor. The wound rotor consists of a laminated iron core with slots carrying the winding, which must have the same number of poles as the stator winding but may have a different number of phases. It is usually wound for three phases and the ends of the windings are brought out to slip rings so that resistances may be inserted in the windings for starting and the terminals short circuited under running conditions.

The squirrel-cage rotor winding consists of a number of heavy copper bars short circuited at the two ends by two heavy brass rings, Fig. 314. The construction is very rugged and there is nothing to get out of order.

When the rotor with its closed windings is placed in the revolving magnetic field produced by the stator currents, the flux cuts across the conductors on the rotor and generates e.m.fs. in them. Currents flow in the rotor equal to the e.m.fs. divided by the rotor

impedances. These currents reacting on the magnetic field produce torque and the rotor revolves in the direction of the field. At no load the rotor runs almost or fast as the field and very small e.m.fs. and currents are induced in its conductors. When the motor is loaded the rotor lags behind the field in speed and large currents are induced to give the required torque.

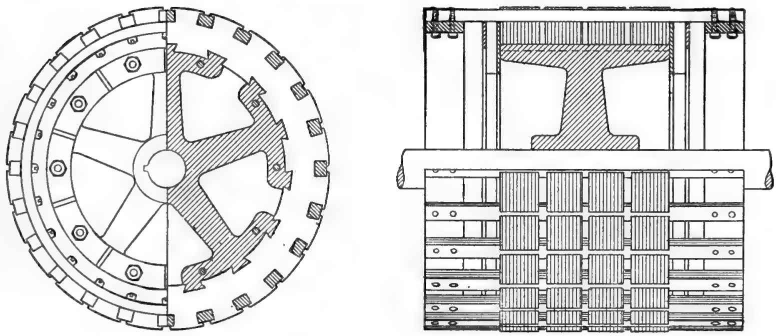


FIG. 314. Squirrel-cage rotor.

204. Slip. The difference between the synchronous speed or speed of the stator field and the speed of the rotor is called the slip and is expressed as a per cent of synchronous speed.

A 6-pole, 60-cycle motor has a synchronous speed

$$N = \frac{120 \times 60}{6} = 1200 \text{ r.p.m.}$$

If the speed of the rotor at full load is 1176 r.p.m., the slip is

$$s = \frac{1200 - 1176}{1200} \times 100 \text{ per cent} = 2 \text{ per cent.}$$

The slip at full load varies from 2 per cent to 5 per cent in motors designed for constant speed.

The rotor speed may be expressed as

$$S = (1 - s)\, N \text{ r.p.m.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (301)$$

205. Magnetomotive Force of the Rotor. The frequency of the e.m.fs. and currents induced in the rotor windings at a slip s is sf if f is the frequency of the e.m.fs. impressed on the stator.

The polyphase currents in the rotor windings produce a resultant m.m.f. revolving relative to the rotor at a speed

$\frac{120\,sf}{p} = sN$ r.p.m. But the rotor itself is revolving at a speed $S = (1 - s)\,N$ r.p.m. and therefore the rotor m.m.f. is revolving at a speed $sN + (1 - s)\,N = N$, that is, at the same speed as the stator m.m.f. The two m.m.fs. are therefore stationary relative to each other and they are nearly opposite in phase as in the transformer.

206. E. M. F. and Flux Diagram for the Induction Motor.

Let r_1 = stator resistance per phase.
L_1 = stator self-inductance per phase.
$x_1 = 2\,\pi f L_1$ = stator reactance per phase.
$z_1 = \sqrt{r_1^2 + x_1^2}$ = stator impedance per phase.
r_2 = rotor resistance per phase.
L_2 = rotor self-inductance per phase.
$x_2 = 2\,\pi f L_2$ = rotor reactance per phase at standstill.
$sx_2 = 2\,\pi s f L_2$ = rotor reactance per phase at slip s.
$z_2 = \sqrt{r_2^2 + x_2^2}$ = rotor impedance per phase at standstill.
$z_2' = \sqrt{r_2^2 + s^2 x_2^2}$ = rotor impedance per phase at slip s.

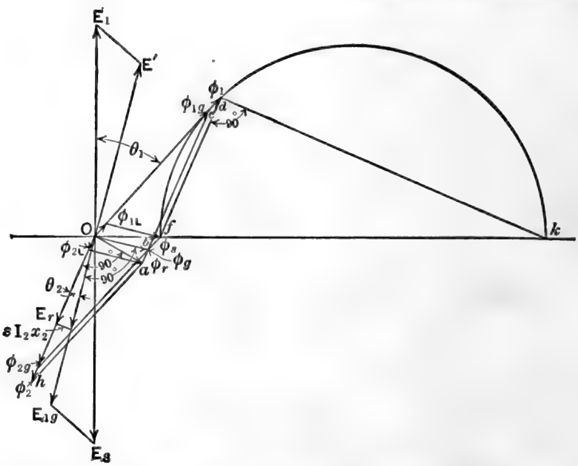


FIG. 315. Flux and e.m.f. diagram of an induction motor.

In Fig. 315

E_1 = the e.m.f. impressed on one phase of the stator.
I_1 = the current in one phase of the stator. $\dot{I}_1 = \dot{I}_M + \dot{I}'$.
I_M = the magnetizing current in one phase of the stator.
I' = the load current in one phase of the stator.

I_2 = the current in one phase of the rotor; $I_2 = \frac{n_1}{n_2} I'$, where

n_1 = the turns per phase per pair of poles on the stator,

n_2 = the turns per phase per pair of poles on the rotor.

Φ_1 = the flux per pole linking one phase of the stator which would be produced by the current I_1 acting alone. $\Phi_1 = \Phi_{1g} + \Phi_{1L}$.

Φ_{1g} = the part of Φ_1 which crosses the gap and links with one phase of the rotor. $\Phi_{1g} = v_1\Phi_1$, where v_1 is a constant.

Φ_{1L} = the part of Φ_1 which does not cross the gap, but is the leakage flux of self-inductance of the stator.

Φ_2 = the flux per pole linking one phase of the rotor, which would be produced by the current I_2 acting alone. $\Phi_2 = \Phi_{2g} + \Phi_{2L}$.

Φ_{2g} = the part of Φ_2 which crosses the gap and links with one phase of the stator. $\Phi_{2g} = v_2\Phi_2$, where v_2 is a constant.

Φ_2 = the part of Φ_2 which does not cross the gap, but is the leakage flux of self-inductance of the rotor.

Φ_g = the actual flux per pole crossing the gap and linking with one phase of both stator and rotor; it is the resultant of Φ_{1g} and Φ_{2g}.

Φ_s = actual flux per pole linking one phase of the stator; it is the resultant of Φ_g and Φ_{1L}.

Φ_r = the actual flux per pole linking one phase of the rotor; it is the resultant of Φ_g and Φ_{2L}.

E_{1g} = the back e.m.f. generated in one phase of the stator by the flux Φ_g.

E_{1L} = the e.m.f. of self-inductance generated in one phase of the stator by the leakage flux Φ_{1L}. $E_{1L} = I_1x_1$.

E_s = the back e.m.f. generated in one phase of the stator by the flux Φ_s. E_s is the resultant of E_{1g} and E_{1L}.

E_1 = the e.m.f. impressed on one phase of the stator. It must be exactly equal and opposite to E_s if the stator resistance drop I_1r_1 is neglected. This drop is of the order of 2 per cent at full load.

Since E_1 is constant, E_s and Φ_s must be constant.

E_{2g} = the e.m.f. generated in one phase of the rotor by the flux Φ_g. $E_{2g} = s\frac{n_2}{n_1}E_{1g} = sE_2$, where E_2 is the e.m.f. which would be generated in one phase of the rotor by the flux Φ_g at standstill.

E_{2L} = the e.m.f. of self-inductance generated in one phase of the rotor by the leakage flux $\Phi_{2L} \cdot E_{2L} = sI_2x_2$.

E_r = the e.m.f. generated in one phase of the rotor by the flux Φ_r. E_r is the difference between E_{2g} and E_{2L} and is equal to I_2r_2. It is in phase with the rotor current I_2 and lags 90 degrees behind the flux Φ_r.

As the induction motor is loaded the end d of the vector Φ_1 follows a circle passing through f and having its centre on of produced.

207. Proof that the Locus is a Circle. From d draw dk at right angles to fd to cut of produced in k. Then the semicircle $f\,dk$ is the locus of d.

In the triangles aob and $f\,dk$

$$\angle oab = \angle f\,dk, \text{ being right angles,}$$
$$< oba = \angle dfk,$$

therefore
$$\frac{fk}{ob} = \frac{fd}{ab} = \frac{fd}{ac - cb} = \frac{fd}{oh - cb},$$

and
$$fk = \frac{ob \times fd}{oh - cb} = \frac{v_1 \cdot of \times v_2 \cdot oh}{oh - v_1v_2oh} = of \cdot \frac{v_1v_2}{1 - v_1v_2} = \text{a constant}$$

since of is constant.

Therefore the locus of d is a circle described on the diameter fk.

208. Magnetomotive Force Diagram. Since the magnetic circuit of the machine is not saturated by the fluxes Φ_s and Φ_r, the flux diagram, Fig. 315, may be replaced by the m.m.f. diagram, Fig. 316.

$od = n_1I_1$ = total m.m.f. of the stator per phase.

$ol = n_1I'$ = m.m.f. of the load component of stator current.

$of = n_1I_M$ = m.m.f. of the magnetizing current.

$op = n_2I_2$ = m.m.f. of the rotor per phase; it is equal and opposite to n_1I'.

209. Stator Current Diagram. The m. m. f. diagram, Fig. 316, may be replaced by the stator current diagram, Fig. 317.

$od = I_1$ = total current in one phase of stator. $\dot{I}_1 = \dot{I}_M + \dot{I}'$.

$of = I_M$ = magnetizing current in one phase of stator.

$ol = I'$ = load component of current in one phase of stator.

$op = I_2$ = current in one phase of rotor. $I_2 = \frac{n_1}{n_2} I'$.

$df = ol = I' = \frac{n_2}{n_1} I_2$ = load component of stator current per phase and represents the rotor current per phase.

To obtain an expression for the diameter of the circle, consider the special case where the motor is at standstill and the rotor and stator resistances are both zero.

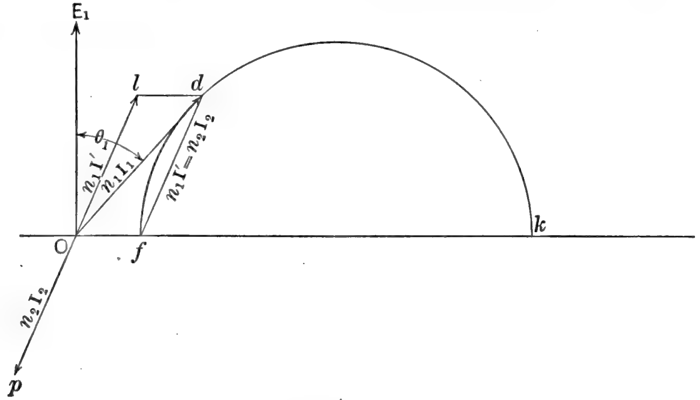


FIG. 316. M.m.f. diagram of an induction motor.

Referring to Fig. 318

$$E_1 = E_s = E_{1g} + I_1 x_1,$$

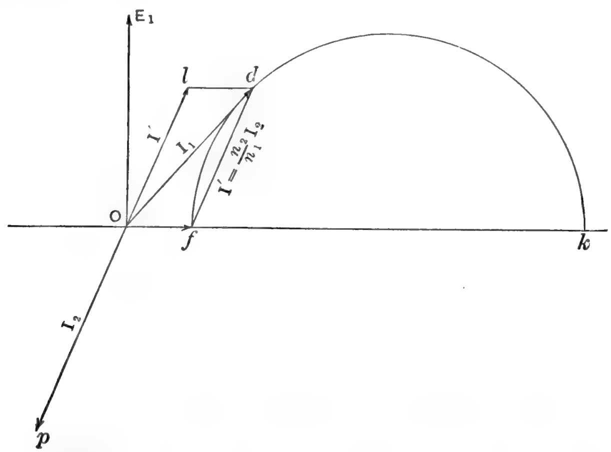


FIG. 317. Current diagram of an induction motor.

and

$$E_{1g} = \frac{n_1}{n_2} E_{2g} = \frac{n_1}{n_2} I_2 x_2 = \left(\frac{n_1}{n_2}\right)^2 I' x_2;$$

therefore,

$$E_{1g} = I_1 x_1 + \left(\frac{n_1}{n_2}\right)^2 I' x_2$$
$$= I'\left(x_1 + \left(\frac{n_1}{n_2}\right)^2 x_2\right),$$

if the magnetizing current I_M is neglected in comparison with the load component of stator current I'.

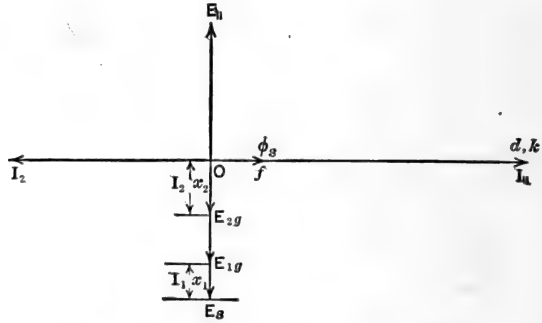


FIG. 318. Induction motor at standstill.

This value of I' is represented by fk and is the diameter of the circle. Its value is

$$D = fk = I' = \frac{E_1}{x_1 + \left(\frac{n_1}{n_2}\right)^2 x_2}. \quad . \quad . \quad . \quad . \quad (302)$$

The diameter of the circle, therefore, varies directly as the impressed e.m.f. and inversely as the sum of the reactances and therefore as the frequency of the supply.

Thus as the motor is loaded the end d of the vector representing the stator current follows a semicircle of diameter

$$\frac{E_1}{x_1 + \left(\frac{n_1}{n_2}\right)^2 x_2}.$$

210. Rotor E. M. F. and Current. The e.m.f. generated in one phase of the rotor at slip s by the flux Φ_g is

$$E_{2g} = sK\Phi_g, \text{ where } K \text{ is a constant,}$$
$$= sE_2, \text{ where } E_2 = K\Phi_g \text{ is the e.m.f.}$$

which would be generated in the rotor at standstill by the flux Φ_g. E_2 does not remain constant as the motor is loaded, since the flux Φ_g does not remain constant but decreases about 30 per cent from no load to standstill when the rotor is locked and slip is unity. Up to full load Φ_g and E_2 may be considered to remain constant.

The impedance of the rotor at slip s is

$$z_2 = \sqrt{r_2^2 + s^2x_2^2}. \qquad (303)$$

The rotor current is

$$I_2 = \frac{sE_2}{z_2} = \frac{sE_2}{\sqrt{r_2^2 + s^2x_2^2}}. \qquad (304)$$

The rotor power factor is

$$\cos\theta_2 = \frac{r_2}{\sqrt{r_2^2 + s^2x_2^2}}. \qquad (305)$$

211. Rotor Input. The power transferred from the stator to the rotor per phase is the product of the back voltage E', generated in the stator by the flux Φ_g, the load component of the stator current I' and the cosine of the angle between them; it is

$$p_r = E'I' \cos\theta_2 \text{ (see Fig. 319)} \qquad (306)$$

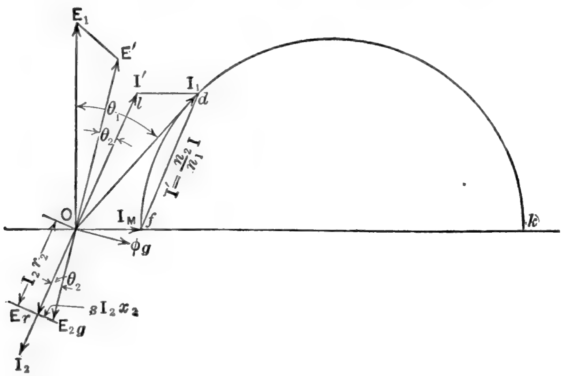


FIG. 319.

but

$$E' = E_{1g} = \frac{n_1}{n_2}E_2\,;$$

and

$$I' = \frac{n_2}{n_1}I_2;$$

therefore,

$$p_r = E'I' \cos\theta_2 = \frac{n_1}{n_2} E_2 \frac{n_2}{n_1} I_2 \cos\theta_2 = E_2 I_2 \cos\theta_2. \quad (307)$$

Thus the power input to the rotor per phase is the product of the e.m.f. which would be generated in the rotor by the flux Φ_g at standstill and the power component of the rotor current.

The total power input to the n phases of the rotor is

$$P_r = nE_2 I_2 \cos\theta_2. \quad (308)$$

212. Rotor Copper Loss and Slip. The power consumed by the rotor copper loss per phase is

$$I_2^2 r_2 = sE_2 I_2 \cos\theta_2, \quad (309)$$

and for the n phases it is

$$L_r = nI_2^2 r_2 = nsE_2 I_2 \cos\theta_2. \quad (310)$$

$$\text{Slip} = \text{the ratio} \frac{\text{rotor copper loss}}{\text{rotor input}} = \frac{L_r}{P_r} = \frac{snE_2 I_2 \cos\theta_2}{nE_2 I_2 \cos\theta_2} = s. \quad (311)$$

213. Rotor Output and Torque. The rotor output per phase is

$$p = p_r - I_2^2 r_2 = E_2 I_2 \cos\theta_2 - sE_2 I_2 \cos\theta_2$$
$$= (1 - s) E_2 I_2 \cos\theta_2. \quad (312)$$

and the total rotor output is

$$P = np = n(1 - s) E_2 I_2 \cos\theta_2 \text{ watts} \quad (313)$$
$$= \frac{n(1 - s) E_2 I_2 \cos\theta_2}{746} \text{ horse power.} \quad (314)$$

From equations 308 and 313 the rotor output is

$$P = (1 - s) P_r = \frac{S}{N} P_r$$

and it is equal to the rotor input multiplied by the rotor speed in per cent of synchronous speed.

If T is the torque in foot pounds the output may be expressed as

$$P = \frac{2\pi ST}{33{,}000} \text{ horse power.}$$

Thus the torque is

$$T = \frac{n(1 - s) E_2 I_2 \cos\theta_2}{746} \times \frac{33{,}000}{2\pi S} = 7.04 \frac{nE_2 I_2 \cos\theta_2}{\dfrac{S}{1 - s}}$$

$$= 7.04 \frac{P_r}{N} = 7.04 \frac{\text{rotor input}}{\text{sync. speed}} \text{ ft. lbs.} \quad (315)$$

The torque of an induction motor is usually expressed not in foot pounds but in synchronous watts, that is, in terms of the power which would be developed at synchronous speed.

The torque in synchronous watts is

$$T_{\text{sync. watts}} = P \times \frac{N}{S} = P_r \quad \ldots \ldots \quad (316)$$

and it is equal to the power input to the rotor.

$$\begin{aligned}
\text{Torque} = T &= 7.04 \frac{\text{rotor input}}{\text{sync. speed}} \\
&= KE_2 I_2 \cos \theta_2 \\
&= KE_2 \cdot \frac{sE_2}{\sqrt{r_2^2 + s^2 x_2^2}} \cdot \frac{r_2}{\sqrt{r_2^2 + s^2 x_2^2}} \\
&= KsE_2^2 \frac{r_2}{r_2^2 + s^2 x_2^2}, \quad \ldots \ldots \quad (317)
\end{aligned}$$

where K is a constant.

The following conclusions may be drawn:

(1) Torque is proportional to the slip near synchronous speed.

(2) Torque is proportional to E_2^2 and therefore it is approximately proportional to the square of the e.m.f. impressed on the stator.

(3) Torque is maximum when $r_2 = sx_2$.

(4) Starting torque $= KE_2^2 \dfrac{r_2}{r_2^2 + x_2^2}$ since $s = 1$.

(5) Maximum torque occurs at standstill if $r_2 = x_2$. This condition can be obtained by inserting resistance in the rotor windings.

If $r_2 > x_2$ the starting torque is less than the maximum torque but the starting current is also less than in the case where $r_2 = x_2$.

(6) The current for maximum torque is

$$\frac{sE_2}{\sqrt{2\, s^2 x_2^2}} = \frac{E_2}{\sqrt{2}\, x_2}$$

and is independent of the rotor resistance.

214. Rotor Efficiency. Neglecting all losses except the rotor copper loss the rotor efficiency is

$$\begin{aligned}
\eta_r = \frac{P}{P_r} 100\% &= \frac{n(1 - s)\, E_2 I_2 \cos \theta_2}{n E_2 I_2 \cos \theta_2} 100\% \\
&= (1 - s) 100\% = \frac{S}{N} 100\%, \quad \ldots \quad (318)
\end{aligned}$$

that is, the rotor efficiency is equal to the rotor speed in per cent of synchronous speed and, therefore, the efficiency of an induction motor is always less than the speed in per cent of synchronous speed.

215. Modification of Diagram. When an induction motor is running without load, a current I_0 flows in each phase of the stator which has two components, Fig. 320, I_M the magnetizing current 90 degrees behind the impressed e.m.f. E_1, and I_P the power component in phase with E_1.

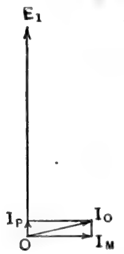


FIG. 320.

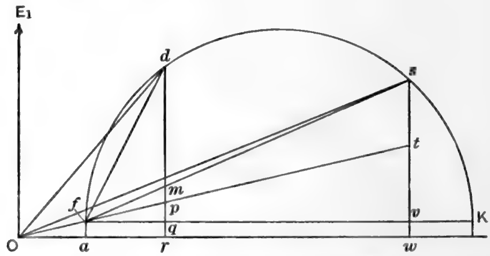


FIG. 321. Circle diagram of an induction motor.

The product nE_1I_P (where n is the number of phases) is the power required to supply the no-load losses. These are the iron loss and a small copper loss in the stator and the friction and windage losses of the rotor. The iron loss in the rotor may be neglected since the rotor frequency is low.

The current required to supply the stator losses has no corresponding component in the rotor, but the power to overcome the friction and windage losses must be transferred from the stator to the rotor and therefore requires a current in the rotor.

As the motor is loaded and slows down the stator iron loss remains nearly constant, the friction and windage losses decrease and the rotor iron loss increases. At standstill the friction and windage losses are absent but the rotor iron loss is large since the rotor frequency is the same as the stator frequency. The iron friction and windage losses are therefore considered to remain constant and the small component of rotor current required to supply the friction and windage losses is neglected.

The diagram, Fig. 317, must therefore be changed to Fig. 321 by the addition of I_P the power component of the stator current per phase at no load.

The diameter of the circle is raised through the distance $af = I_P$ and of now represents not the magnetizing current I_M but the no-load current $I_0 = \sqrt{I_M^2 + I_P^2}$.

If os represents the stator current per phase at standstill and sw is its power component, then, since there is no output, the power input is consumed by the losses. Therefore, input = losses $= nE_1sw$; constant losses $= nE_1I_P = nE_1vw$ and copper losses $= nE_1sv = n\,(I'^2r_1 + I_2^2r_2)$.

The stator copper loss is taken as nI'^2r_1 because the stator copper loss at no load is included in the constant losses. It is therefore assumed that $I_1^2r_1 = I_0^2r_1 + I'^2r_1$ which is approximately correct up to full load.

Divide sv at t so that $st : tv = I_2^2r_2 : I'^2r_1$, then, stator load copper loss $= nE_1tv$ and rotor copper loss $= nE_1st$.

Join ft and from d any point on the circle to the left of s draw $dmpqr$ perpendicular to the diameter fk. It is to be shown that mp is the stator current required to supply the rotor copper loss for a rotor current represented by fd and that pq is the current to supply the corresponding stator load copper loss.

The rotor copper loss is

$$nI_2^2r_2 = n\left(\frac{n_1}{n_2}I'\right)^2 r_2 = KI'^2 = K\overline{fd}^2 = K\,(\overline{fq}^2 + \overline{qd}^2)$$

$$= K \times fq\,(fq + qK) = K \times fq \times D = fq \times \text{a constant},$$

since K and D are both constants.

Therefore the rotor copper loss is proportional to fq; but

$$\frac{mp}{fq} = \frac{st}{fv}, \text{ or } \frac{mp}{(fd)^2} = \frac{st}{(fs)^2},$$

and since st represents the rotor copper loss for a current fs, mp represents the rotor copper loss for a current fd.

Similarly pq represents the stator copper loss for stator current fd.

216. Interpretation of Diagram. At any value of stator current $od = I_1$, Fig. 321,

$nE_1\,dr$ = stator input in watts,
nE_1qr = constant losses,
nE_1pq = stator copper loss,
nE_1mp = rotor copper loss,
$nE_1\,dm$ = rotor output in watts = mechanical load,

$$\frac{dr}{od} = \text{power factor},$$

$$\frac{dm}{dr} = \text{efficiency},$$

$$\frac{mp}{dp} = \frac{\text{rotor copper loss}}{\text{rotor input}} = \text{slip},$$

$$\frac{dm}{dp} = \frac{\text{rotor output}}{\text{rotor input}} = \frac{\text{actual speed}}{\text{synchronous speed}} = \frac{S}{N} = 1 - s.$$

The torque corresponding to output $nE_1\,dm$ is

$$T = \frac{nE_1\,dm}{746} \times \frac{33{,}000}{2\,\pi\,(\text{r.p.m.})}\ \text{lbs. at 1 ft. radius.}$$

At synchronous speed this torque would represent an output

$$nE_1\,dm \times \frac{N}{S} = nE_1\,dm \times \frac{dp}{dm} = nE_1\,dp \text{ watts} = \text{rotor input.}$$

The torque in synchronous watts is equal to the watts input to the rotor $= nE_1\,dp$.

At standstill the torque in synchronous watts is nE_1st and represents the starting torque of the motor.

The maximum value of torque in synchronous watts is $nE_1 \times$ maximum value of dp.

The maximum output in watts is $nE_1 \times$ maximum value of dm. For average 25-cycle motors, starting torque is $1\frac{1}{2}$ to $2\frac{1}{2}$ times full-load torque; starting current is 6 to 8 times full-load current; and maximum running torque is $2\frac{1}{2}$ to $3\frac{1}{2}$ times full-load torque. For 60-cycle motors, starting torque is 1 to $1\frac{1}{2}$ times full-load torque; starting current is 5 to 6 times full-load current; and maximum running torque is 2 to $2\frac{1}{2}$ times full-load torque.

217. Construction of Diagram from Test for a Three-phase Motor. 1. Run the motor light at rated voltage and rated frequency. Read impressed voltage, current and watts input E_1, I_0 and W_0.

$$I_0 = of \text{ on the diagram.}$$

$$\frac{W_0}{\sqrt{3}\,E_1} = I_P = af = wv \text{ on the diagram.}$$

2. Lock the rotor and impress reduced voltage and raise it until twice full-load current flows in the stator. Read impressed voltage, current, watts input and torque, E_L, I_L, W_L and T_L. To

get the value of locked current at rated voltage, raise the values of I_L in the ratio $E_1 : E_L$. To get the values of locked watts and locked torque at rated voltage, raise the values of W_L and T_L in the ratio $E_1^2 : E_L^2$. To get accurate results for the circle diagram it is better to reduce the values of watts and torque to terms of power current per phase. This is done by dividing the values of W_L and T_L by $\sqrt{3}\, E_L$.

Plot on a base of impressed voltage,

$$(1)\ I_L, \quad (2)\ \frac{W_L}{\sqrt{3}\, E_L}, \quad (3)\ \frac{T_L}{\sqrt{3}\, E_L}.$$

These three loci should be straight lines passing through the origin and can be produced till they cut the ordinate at the rated voltage of the motor.

The following results are obtained.

Value of I_L at rated voltage $= os$ on the diagram.

Value of $\dfrac{W_L}{\sqrt{3}\, E_L}$ at rated voltage $= sw$ on the diagram.

3. Measure the resistance of the stator per phase $= r_1$. Then the stator copper loss locked at rated voltage is $3\ \overline{os}^2 r_1$ watts and $\dfrac{3\ \overline{}^2 r_1}{\sqrt{3}\, E_1} = vt$ on the diagram.

The rotor copper loss, locked at rated voltage, is $\sqrt{3}\, E_1 \times st$ and is known since $st = sw - wv - vt$.

The rotor copper loss also represents the starting torque in synchronous watts and therefore if T_1 is the value of T_L at rated voltage

$$\frac{T_1 \times 2\pi \times (\text{sync. speed r.p.m.})}{33{,}000} \text{ should equal } \frac{\sqrt{3}\, E_1 \times st}{746}.$$

The circle diagram can be drawn in from the values obtained above.

The motor has been assumed to be Y-connected and the voltage E_1 is the line voltage.

218. Methods of Starting. Except in the case of small machines, induction motors should not be started by connecting them directly to the mains, since the large starting current at low power factor disturbs the voltage regulation of the system.

Two methods of reducing the starting current are in use. (1) The voltage impressed on the stator is reduced by multi-voltage taps on the supply transformers or by using a potential starter. (2) Resistance is inserted in series with the rotor windings.

(1) When the impressed voltage is reduced, the starting current is reduced in proportion to it, but the starting torque is reduced as the square of the voltage.

In Fig. 322 $E_1.s_1t_1$ represents the starting torque of the motor

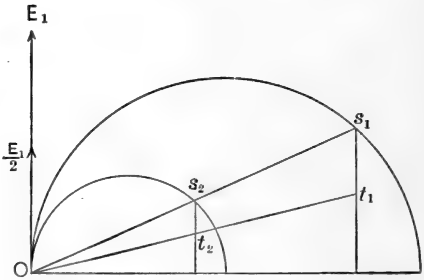


FIG. 322. Starting at reduced voltage.

at full voltage and os_1 represents the starting current, neglecting the exciting current, and $\frac{E_1}{2} \cdot s_2t_2$ represents the starting torque at half voltage and os_2 represents the starting current.

Since $os_2 = \frac{os_1}{2}$ the starting current is reduced to one half its value at full voltage, but the starting torque is reduced to one quarter. The power factor is not changed.

Thus starting with reduced voltage gives very small starting torque and low power factor.

A squirrel-cage rotor may be used.

(2) When resistance is inserted in the rotor windings the starting current is reduced and is brought more nearly in phase and the starting torque is increased.

In Fig. 323 s_0t_0 represents the starting torque when the rotor circuits are closed without any starting resistance.

s_1t_1 is the starting torque when resistance R_1 is inserted.

s_2t_2 is the starting torque when resistance $R_2 > R_1$ is inserted.

s_3t_3 is the maximum possible starting torque and is obtained by inserting a resistance $R_3 > R_2$; it is the same as the maximum

running torque of the motor. $R_3 + r_2 = x_2$, where r_2 and x_2 are the resistance and the reactance of the rotor. os_0, os_1, os_2 and os_3 are the corresponding stator currents and cos θ_0, cos θ_1, etc., are the power factors at start.

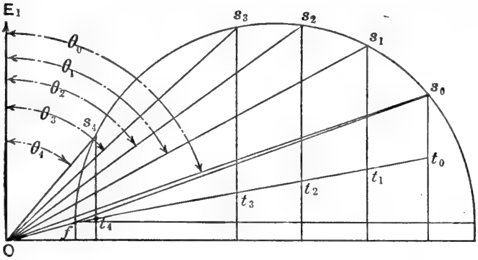


FIG. 323. Starting torque with various rotor resistances.

The curves in Fig. 324 are the "speed-torque" characteristics for the motor operating with the various resistances in the rotor. The maximum torque is the same in all cases but it is reached at different speeds. One current curve holds in all cases.

If a resistance $R_4 > R_3$ is inserted in the rotor windings the starting current is further reduced and the power factor is improved but the starting torque is decreased. R_4 may be made of such value that the starting torque s_4t_4 is equal to full-load torque and the starting current os_4 is equal to full-load current. Curve (4) is the speed-torque characteristic for this case.

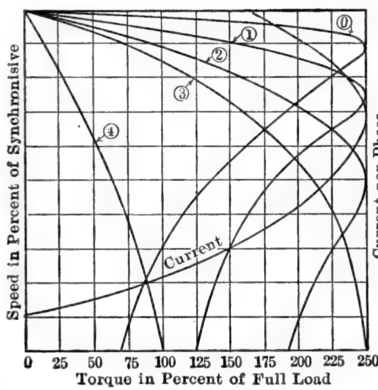


FIG. 324. Speed-torque characteristics of an induction motor with various rotor resistances.

Thus by inserting resistance in the rotor any starting torque up to the maximum running torque or "pull out" torque may be obtained. The starting current is reduced and the power factor is improved.

In starting a heavy load resistance R_3 is used and the motor gives its maximum torque at start. The resistance is then cut out gradually as the speed increases and the motor operates with short-circuited rotor with characteristics as shown in curve (0).

If the load to be started is not very great and a large starting current at low power factor is objectionable, resistance R_4 is used and the motor starts with full-load torque and draws full-load current.

This second method of starting requires a wound rotor with slip rings and large starting resistances which is much more expensive than a squirrel-cage rotor.

For the same line current, resistance starting gives about four times the torque given at reduced voltage.

219. Applications. The constant-speed or squirrel-cage induction motor takes the place of the direct-current shunt motor and has very similar characteristics. It is of much more simple and rugged construction than the shunt motor and the wear and danger due to sparking is entirely eliminated.

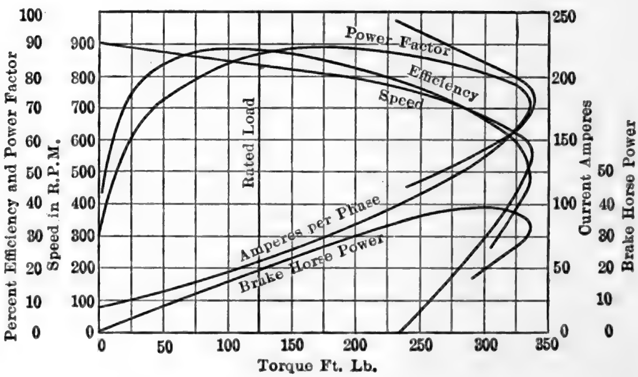


FIG. 325. Characteristic curves of a three-phase, 60-cycle, 220-volt, 20-horse-power induction motor.

It should be used where fairly constant power is required for long periods, where good speed regulation is required, where starting is infrequent and only average starting torque is necessary, where the motor is exposed to dust or to inflammable materials or is not easily inspected. It is suitable for driving line shafting, for high- and low-speed centrifugal pumps, blowers, fans, etc. It must be started on reduced voltage except for the smallest sizes.

The variable-speed induction motor has a wound rotor with its terminals connected to slip rings so that resistance may be introduced to vary the speed or to give a large starting torque.

It should be used where frequent starts under load are necessary, or where the motor is large enough to have a bad effect on the regulation of the system due to the large starting current at low power factor, as for cranes, elevators, hoists, etc.

The squirrel-cage motor with a comparatively high resistance rotor can be used where fairly large starting torque is required and where a wound-rotor motor is not advisable as, in cement mills, etc.

Fig. 325 shows the characteristic curves of a three-phase, 60-cycle, 220-volt, 20-horse-power induction motor with a squirrel-cage rotor.

220. Speed Control of Induction Motors. The induction motor is inherently a constant-speed motor. It has been seen above that the speed can be varied by using a wound rotor and connecting resistances in series with the rotor windings, but this is a very wasteful method since the efficiency of a motor is always less than its speed in per cent of synchronous speed.

To vary the speed efficiently the synchronous speed must be varied. This can be accomplished by means of special windings arranged with different numbers of poles. Such windings are very complicated and expensive and the number of speeds is limited to three. A squirrel-cage rotor must be used.

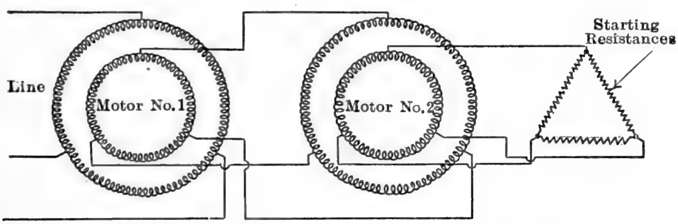


FIG. 326. Cascade control of induction motors.

A second method of varying the synchronous speed is by concatenation or cascade control. Two similar motors with wound rotors are rigidly connected to the same shaft. The stator of the first motor is connected to the line; the stator of the second motor is connected to the rotor winding of the first motor and receives power from it; the rotor of the second motor is closed through starting resistances. (Fig. 326.)

The frequency of the e.m.fs. generated in the rotor of an induc-

tion motor is sf, where f is the frequency of the supply and s is the slip. Thus the frequency impressed on the stator of the second motor is the frequency of slip of the first motor. The speed of the two motors is always the same and thus at no load $(1 - s)f = sf$ and $s = 0.5$.

Therefore, two similar motors connected in cascade tend to approach a speed of half synchronous speed at no load and fall below this speed under load.

Speeds below half synchronous speed are obtained by inserting resistance in the rotor windings of the second motor.

For speeds above half synchronous speed the stator of the second motor must be connected to the line and the rotor of the first motor closed through resistances.

This method of control is used for some three-phase traction systems and is very similar to the series-parallel control of direct-current series motors. The induction motor, however, does not tend to increase its speed indefinitely and if it operates above synchronous speed it acts as a brake and pumps back power into the lines.

221. Analysis by Rectangular Coördinates. Using rectangular coördinates the performance characteristics of an induction motor can be determined if the constants of the motor are known.

Let

$y = g - jb$ = stator exciting admittance per phase, measured with the rotor circuits open so that the friction losses are not included. The rotor must be driven at synchronous speed.

$\dot{z}_1 = r_1 + jx_1$ = stator impedance per phase.

$\dot{z}_2 = r_2 + jsx_2$ = rotor impedance per phase at slip s.

Assume that the ratio of turns is $n_1 : n_2 = 1 : 1$ and take as real axis of coördinates the e.m.f. generated in the stator by the flux of mutual inductance. The quantities used refer to one phase of the stator and the corresponding phase of the rotor.

E' = e.m.f. generated in the stator.

$E_2 = E'$ = e.m.f. generated in the rotor at standstill.

sE_2 = e.m.f. generated in the rotor at slip s.

These three e.m.f.'s are written as absolute values without the dot since they lie along the axis.

The following equations show the relations between the various e.m.f.'s and currents at slip s.

Rotor current

$$\dot{I}_2 = \frac{sE_2}{r_2 + jsx_2} = E_2\left(\frac{sr_2}{r_2^2 + s^2x_2^2} - j\frac{s^2x_2}{r_2^2 + s^2x_2^2}\right) = E'\,(a_1 - ja_2). \qquad (319)$$

Stator load current

$$\dot{I}' = \dot{I}_2 = E'\,(a_1 - ja_2). \quad \dots \dots \quad (320)$$

Stator exciting current

$$\dot{I}_e = E'\dot{y} = E'\,(g - jb). \quad \dots \dots \quad (321)$$

Total stator current

$$\dot{I}_1 = \dot{I}' + \dot{I}_e = E'\,\{(a_1 + g) - j\,(a_2 + b)\} = E'\,(b_1 - jb_2). \quad (322)$$

E.m.f. impressed on the stator

$$\begin{aligned} \dot{E}_1 &= E' + \dot{I}_1\dot{z}_1 = E' + E'\,(b_1 - jb_2)\,(r_1 + jx_1) \\ &= E'\,\{(1 + b_1r_1 + b_2x_1) + j\,(b_1x_1 - b_2r_1)\} \\ &= E'\,(c_1 + jc_2); \quad \dots \dots \dots \quad (323) \end{aligned}$$

its absolute value which remains constant is

$$E_1 = E'\,\sqrt{c_1^2 + c_2^2}. \quad \dots \dots \quad (324)$$

Thus the e.m.f. generated in the stator is

$$E' = \frac{E_1}{\sqrt{c_1^2 + c_2^2}}. \quad \dots \dots \quad (325)$$

Substituting this value for E' in the equations above the absolute values of the various quantities at slip s can be obtained.

Rotor current

$$I_2 = I' = E'\,\sqrt{a_1^2 + a_2^2} = E_1\frac{\sqrt{a_1^2 + a_2^2}}{\sqrt{c_1^2 + c_2^2}}. \quad \dots \quad (326)$$

Exciting current

$$I_e = E'\,\sqrt{g^2 + b^2} = E_1\frac{\sqrt{g^2 + b^2}}{\sqrt{c_1^2 + c_2^2}}. \quad \dots \dots \quad (327)$$

Stator current

$$I_1 = E'\,\sqrt{b_1^2 + b_2^2} = E_1\frac{\sqrt{b_1^2 + b_2^2}}{\sqrt{c_1^2 + c_2^2}}. \quad \dots \dots \quad (328)$$

The torque in synchronous watts was found in Art. 213 to be equal to the rotor input in watts which is

$$\begin{aligned} P_r &= nE_2I_2 \cos \theta_2 \\ &= nE' \times E'a_1 = nE'^2a_1 \\ &= nE_1^2 \frac{a_1}{c_1^2 + c_2^2} = nE_1^2 \frac{sr_2}{(c_1^2 + c_2^2)(r_2^2 + s_2^2x_2^2)}, \quad (329) \end{aligned}$$

where n is the number of phases.

The rotor output neglecting friction losses is

$$\begin{aligned} P &= (1 - s)\, Pr \\ &= nE_1^2 \frac{sr_2 (1 - s)}{(c_1^2 + c_2^2)(r_2^2 + s^2x_2^2)}. \quad (330) \end{aligned}$$

The stator power factor is $\cos \theta_1$, where θ_1 is the angle of lag of the current I_1 behind the impressed e.m.f. E_1. $\theta_1 = \theta' + \theta''$,

$$\cos \theta' = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \cos \theta'' = \frac{b_1}{\sqrt{b_1^2 + b_2^2}};$$

therefore

$$\begin{aligned} \cos \theta_1 &= \cos (\theta' + \theta'') = \cos \theta' \cos \theta'' - \sin \theta' \sin \theta'' \\ &= \frac{c_1b_1 - c_2b_2}{\sqrt{c_1^2 + c_2^2}\sqrt{b_1^2 + b_2^2}}. \end{aligned}$$

The power input to the stator is

$$\begin{aligned} P_1 &= nE_1I_1 \cos \theta_1 \\ &= nE_1 \times E_1 \frac{\sqrt{b_1^2 + b_2^2}}{\sqrt{c_1^2 + c_2^2}} \times \frac{c_1b_1 - c_2b_2}{\sqrt{c_1^2 + c_2^2}\sqrt{b_1^2 + b_2^2}} \\ &= nE_1^2 \frac{c_1b_1 - c_2b_2}{c_1^2 + c_2^2}. \quad (331) \end{aligned}$$

Using these equations the various quantities can be calculated in terms of the slip and the characteristic curves of the motor plotted.

222. Single-phase Induction Motor. The stator of a single-phase induction motor has a single winding with any number of pairs of poles.

The rotor is either of the squirrel-cage type or is wound with the same number of poles as the stator but with any number of phases.

Fig. 327 shows the relative directions of the currents in the two windings at standstill. The stator carries a current I_1 which consists of two components, I' the load component and I_M the magnetizing current. The rotor carries a current I_2 opposite in phase to I' and equal to it in m.m.f. If the ratio of turns is assumed to be $n_1 : n_2 = 1 : 1$; then $I_2 = I'$. The motor at standstill is a transformer with a short-circuited secondary.

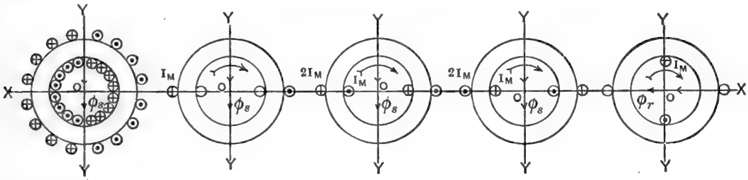


FIG. 327. FIG. 328. FIG. 329. FIG. 330. FIG. 331.
Single-phase induction motor.

The flux which crosses the air gap and links with both stator and rotor is produced by the stator exciting current. It is always directed along the line YOY. There is no component of flux in the horizontal direction XOX and therefore no torque is exerted tending to turn the rotor in either direction. Thus the rotating field which is produced in the polyphase induction motor does not exist in the single-phase motor at standstill. The single-phase induction motor, therefore, has no starting torque. If, however, it is started in either direction it will develop torque and will accelerate and come up approximately to synchronous speed at no load.

Fig. 328 represents the motor with the rotor open circuited and, therefore, without current in its windings. The stator carries only the magnetizing current.

Fig. 329 represents conditions at synchronous speed at the instant when the stator magnetizing current is maximum. The stator flux is then maximum downwards.

The rotor conductors moving at synchronous speed cut the stator flux and an e.m.f. is generated in them proportional to the product of flux and speed. Since the flux is alternating the e.m.f. generated is of double frequency and produces a current of double frequency in the closed rotor winding. The current produces a flux the rate of change of which through the rotor windings generates in them an e.m.f. equal and opposite to the e.m.f. generated

by rotation. This flux must, therefore, be of the same value as the stator flux and it is in phase with the rotor current.

The rotor current goes through two complete cycles during one revolution. In Fig. 329 it is maximum and is opposed to the stator current, but the e.m.f. impressed on the stator is constant and the stator flux is constant, and, therefore, a current must flow in the stator to balance the m.m.f. of the rotor current I_M'. Since the ratio of turns has been taken as 1 : 1 the increase in stator current is I_M' and the total stator current at synchronous speed is $I_M + I_M'$. In the position shown the rotor flux is not produced because the rotor m.m.f. is opposed by an equal and opposite m.m.f. on the stator.

Fig. 330 represents conditions after the rotor has turned through one half a revolution and the stator current has passed through one half cycle. The rotor current is in the same direction as before and has completed one cycle.

Fig. 331 represents conditions midway between Fig. 329 and Fig. 330. The stator current is zero and the rotor current is maximum and exerts a m.m.f. in the horizontal direction. There is no stator m.m.f. opposing it and a flux is produced of the same value as the stator flux in Fig. 329 or Fig. 330. Since the reluctance of the path for the horizontal flux is the same as that for the vertical stator flux, the rotor magnetizing current I_M' must be equal to the stator magnetizing current at standstill I_M, and, therefore, at synchronous speed the stator magnetizing current is $2\,I_M$ and is double its value at standstill.

Thus at synchronous speed there is a resultant m.m.f. of constant value revolving at synchronous speed and the magnetic field of the single-phase motor is identical with that of the polyphase motor, Fig. 332. The m.m.f. to produce the vertical field is supplied by the true stator magnetizing current, while the m.m.f. to produce the horizontal field is provided by an equal stator magnetizing current, in phase with the true stator magnetizing current, which induces in the rotor the rotor magnetizing current.

When the rotor runs at synchronous speed its conductors do not cut this revolving flux and the only current in the rotor is the double-frequency magnetizing current.

When the rotor runs at a slip s below synchronous speed the rotor conductors cut the flux and currents are produced in them and torque is developed just as in the case of the polyphase motor.

223. Horizontal Field at Slip s. When the rotor runs at a speed $S = (1 - s) \times$ synchronous speed, the e.m.f. generated in it due to cutting the stator flux is less than at synchronous speed in the ratio $1 - s : 1$ and the horizontal flux and the rotor magnetizing current are less in the same ratio.

The stator current is $\dot{I}_M + (1 - s)\ \dot{I}_M + \dot{I}'$ and the rotor current is $(1 - s)\ \dot{I}_M + \dot{I}_2$. The frequency of the rotor magnetizing current is $(2 - s)f$ and the frequency of the rotor load current is sf, where f is the frequency of the e.m.f. impressed on the stator.

The revolving field at slip s is not constant in value but has the horizontal axis shorter than the vertical in the ratio $1 - s : 1$, Fig. 333. The field follows an elliptical instead of a circular locus.

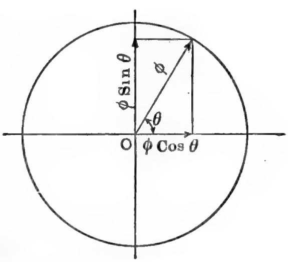


FIG. 332. Revolving field of a single-phase induction motor at synchronous speed.

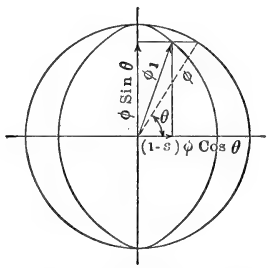


FIG. 333. Revolving field of a single-phase induction motor at slip s.

The torque which is proportional to the product of the rotor load current and the horizontal field is less than that produced in the polyphase motor in the ratio $1 - s : 1$.

224. Starting Single-phase Induction Motors. In order to obtain the torque required to start a single-phase induction motor a component of flux in quadrature in time and in space with the stator flux must be produced at standstill. It has been shown that when once the motor is started the rotor produces the required quadrature flux and thus the torque to carry the load.

Two principal methods are employed to produce the quadrature flux at standstill, (1) phase splitting and (2) shading coils.

(1) If the two stator windings of a two-phase induction motor are connected to a single-phase supply, phase 1 directly and phase 2 through a suitable resistance or condensive reactance, the flux produced by phase 2 will have a component in quadrature in time

with phase 1 and will thus give the required starting torque. (Fig. 334.) When the motor has come up to half speed, the starting winding is cut out and the motor runs as a single-phase motor on

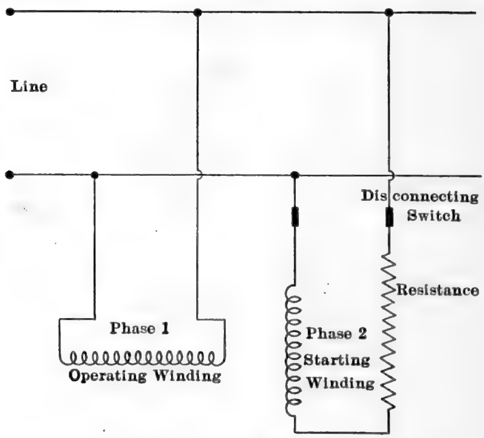


FIG. 334. Phase splitting.

phase 1. This method of starting is called phase splitting. The second winding need not have as many turns as the first but it should be placed at 90 electrical degrees to it.

(2) The shading coil, Fig. 335, is a short-circuited coil surrounding part of each pole of the stator. Currents are induced in it and oppose the increase and decrease of the flux in the part of the pole which it incloses. Thus the north pole in section A will reach its maximum value before it is maximum in section B. When the north pole has decreased to zero in B it will be increasing in section C and thus there is a rotation of the magnetic field and torque is produced. When the motor is started the short-circuited coils are opened and they are then idle.

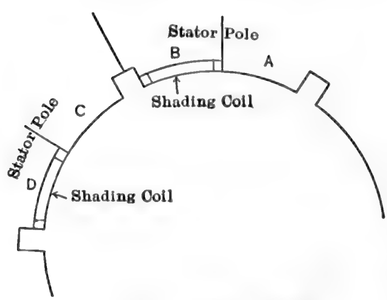


FIG. 335. Shading coils.

225. Comparison of Single-phase and Polyphase Motors. Take the case of a two-phase motor operating on a single-phase circuit using only one phase of the stator winding.

The slip single-phase is less than two-phase since the whole rotor corresponds to one phase of the stator and thus the rotor current and rotor copper loss are decreased.

The efficiency is lower because the output decreases more than the losses. For a given impressed e.m.f. and frequency the iron and friction losses remain practically constant.

The power factor is lower because the magnetizing current is approximately doubled.

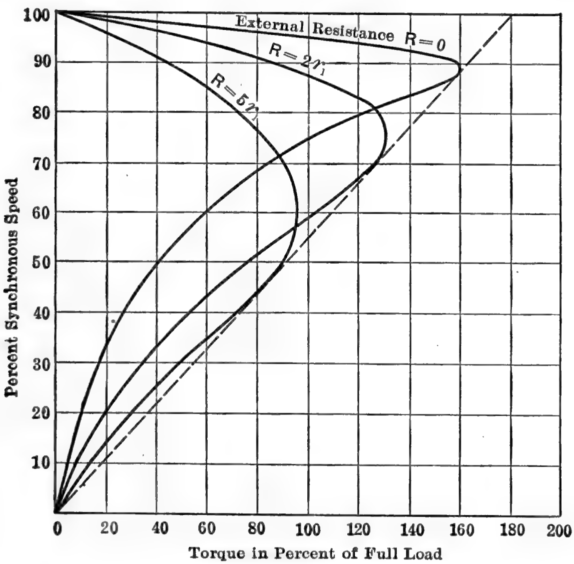


FIG. 336. Speed-torque characteristics of a single-phase induction motor with various resistances inserted in the rotor.

A given motor wound single-phase can be operated at higher densities than when wound polyphase since the losses are less and its ventilation is the same, and in this way its output may be made from 65 to 75 per cent of its output polyphase.

The torque at any speed can be increased by introducing resistance into the rotor windings, but this changes the maximum torque since the torque is proportional to $1 - s$.

Fig. 336 shows typical speed-torque curves of a single-phase induction motor with various external resistances connected in the rotor windings.

A single-phase induction motor is usually either a two-phase or three-phase motor operated on a single-phase circuit using only part of the stator winding.

If one phase of a two-phase motor is opened at light load, the magnetizing current of the other phase is doubled and the motor runs as a single-phase motor. If two phases of a three-phase motor are opened the motor runs as a single-phase motor with the magnetizing current in the third phase trebled. In both cases the flux distribution and flux densities remain approximately the same as before.

226. Induction Generator. If the stator of an induction motor is connected to the supply lines and its rotor is driven above synchronous speed, the machine will develop electrical power and supply it to the system.

The stator flux is not affected by the increase in the speed of the rotor, but revolves in the same direction as when the machine operates as a motor. The slip is, however, reversed and the e.m.fs. and currents induced in the rotor are reversed. Thus the direction of torque and power is reversed and the mechanical power supplied to drive the rotor is transformed into electrical power and supplied over the lines to the load.

The power transferred from the rotor to the stator depends on the slip just as in the induction motor. Using the same notation as in Art. 211, the power transferred to the stator is

$$\begin{aligned} P &= nE_2I_2 \cos \theta_2 \\ &= nE_2 \frac{sE_2}{\sqrt{r_2^2 + s^2x_2^2}} \cdot \frac{r_2}{\sqrt{r_2^2 + s^2x_2^2}} \\ &= \frac{snE_2^2r_2}{r_2^2 + s^2x_2^2}. \end{aligned}$$

Thus to increase the power delivered by the generator its speed must be increased. If therefore an induction generator is connected to a prime mover of variable speed, it will supply power almost in proportion to the increase of its speed above synchronous speed.

The frequency of the stator induced e.m.f. is the frequency of the exciting current and does not depend in any way on the speed at which the rotor is driven.

The induction generator has two very serious disadvantages, it is not self-exciting and it cannot supply wattless currents to an

inductive load. It must therefore be operated in parallel with an alternator of sufficient capacity to supply both the wattless current to excite the induction generator and also the wattless current required by the load. (Fig. 337.) Since the exciting current is about 30 per cent of full-load current the induction generator is not suitable when the load power factor is low.

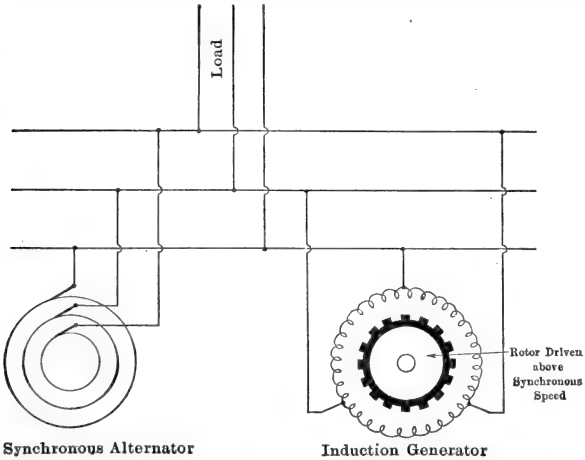


FIG. 337. Induction generator.

The alternator which provides the exciting current for the induction generator and thus determines the frequency of the system may be located in the receiving station and will then supply the wattless exciting current back over the line. This tends to improve the power factor of the line and the voltage regulation since a lagging current supplied from the receiver end to the generating end is equivalent to a leading current in the opposite direction.

In construction the induction generator is the same as an induction motor and a squirrel-cage rotor may be used.

CHAPTER VIII

ALTERNATING-CURRENT COMMUTATOR MOTORS

227. Alternating-current Series Motor. The alternating-current series motor is very similar to the direct-current series motor and can be operated on direct-current with increased efficiency and output.

If a direct-current series motor is connected to an alternating-current supply circuit it will rotate since the currents in the field and armature reverse together and therefore the torque is always in one direction, but it will be very inefficient and will spark very badly.

With alternating current flowing in the field winding an alternating magnetic flux is set up through the magnetic circuit and causes very large losses due to hysteresis and eddy currents. To reduce these to a minimum the whole magnetic circuit of an alternating-current series motor must be laminated. The field circuit must be very heavily insulated to prevent short circuits between turns which would burn out the motor on account of the large induced currents.

The relation between the e.m.fs. and current in the direct-current series motor is given by the equation

$$E = \mathcal{E} + I\,(r_a + r_f), \quad . \quad . \quad . \quad . \quad . \quad (332)$$

where E = impressed e.m.f.,

$\mathcal{E}$ = counter e.m.f. generated by rotation,

I = current,

r_a = resistance of the armature,

r_f = resistance of the field.

In the alternating-current series motor the alternating flux sets up large e.m.fs. of inductance in both the field and armature windings, which consume components of the impressed e.m.f. in quadrature ahead of the current. If L_f is the inductance of the field and L_a the inductance of the armature, their reactances are

$x_f = 2\pi f L_f$ and $x_a = 2\pi f L_a$, respectively, where f is the frequency of the impressed e.m.f.

Fig. 338 shows the vector diagram for the motor.

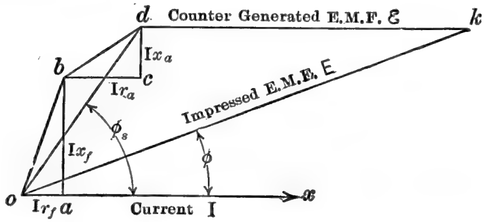


FIG. 338. Vector diagram of a single-phase series motor.

$ox = I$ = current in field and armature.
$oa = Ir_f$ = e.m.f. consumed by the resistance of the field.
$ab = Ix_f$ = e.m.f. consumed by the reactance of the field.
$bc = Ir_a$ = e.m.f. consumed by the resistance of the armature.
$cd = Ix_a$ = e.m.f. consumed by the reactance of the armature.
$dk = \mathcal{E}$ = e.m.f. generated in the armature due to rotation, in phase with the field flux and, therefore, in phase with the current, neglecting the hysteretic lag.
$ok = E$ = impressed e.m.f.
$\cos kox = \cos \phi$ = load power factor.
$\cos dox = \cos \phi_s$ = power factor at start.

Taking the current as the real axis the relation between the current and the impressed e.m.f. can be expressed in rectangular coördinates as

$$\dot{E} = \dot{\mathcal{E}} + I(r_a + r_f) + jI(x_a + x_f) \quad . \quad . \quad . \quad (333)$$

and taking absolute values

$$E = \sqrt{\{\mathcal{E} + I(r_a + r_f)\}^2 + \{I(x_a + x_f)\}^2}. \quad . \quad . \quad (334)$$

At standstill

$$E = I\sqrt{(r_a + r_f)^2 + (x_a + x_f)^2} \quad . \quad . \quad . \quad (335)$$

and the current is

$$I = \frac{E}{\sqrt{(r_a + r_f)^2 + (x_a + x_f)^2}}. \quad . \quad . \quad . \quad . \quad . \quad (336)$$

Full voltage can usually be impressed on the motor at standstill without causing any injury since the current is limited by the large impedance.

The power factor under running conditions is

$$\cos \phi = \frac{\mathcal{E} + I(r_a + r_f)}{\sqrt{\{\mathcal{E} + I(r_a + r_f)\}^2 + \{I(x_a + x_f)\}^2}} \quad . \quad (337)$$

but $\mathcal{E} = kn\Phi$, where Φ is the maximum value of the flux per pole, n is the motor speed in revolutions per second and k is a constant depending on the number of turns in the armature winding and on the shape of the flux wave. The flux Φ is almost proportional to the current I and the generated e.m.f. may be expressed as

$$\mathcal{E} = k'nI.$$

Substituting this value for $\mathcal{E}$ in equation and eliminating I

$$\cos \phi = \frac{k'n + r_a + r_f}{\sqrt{(k'n + r_a + r_f)^2 + (x_a + x_f)^2}}; \quad . \quad . \quad (338)$$

the power factor, therefore, increases with increasing speed and approaches unity. At low speed and at standstill it is low on account of the reactances in the field and armature and for satisfactory operation it is necessary to make these reactances as low as possible.

228. Design for Minimum Reactance. The inductance of any coil is proportional to the square of the number of turns and is inversely proportional to the reluctance of the magnetic circuit through it. To reduce the inductance L_f of the field winding it is designed with a small number of turns but this reduces the field m.m.f. and in order to obtain the required flux the reluctance of the magnetic circuit must be made very low. For this purpose large sections of high permeability are used, the slots are partially closed and the air gap is made as short as possible.

The reactance of the winding is proportional to the product of the inductance and the frequency and therefore the frequency should be low. Motors are usually designed for 25 cycles since that is the lowest standard frequency, but they will operate on 15 cycles or on direct current with a much improved efficiency and power factor and a larger output. The frequency of the supply does not affect the speed of the motor directly, but it does indirectly since the reactance drop decreases with the frequency and, therefore, the speed for a given current increases.

229. Compensating Windings. The armature inductance and reactance cannot be decreased by reducing the number of

turns on the armature since, for a given impressed voltage, that would increase the speed of the motor and, further, since the field is made comparatively weak the armature must be made correspondingly strong in ampere turns in order to produce the required torque.

The armature m.m.f. as in direct-current machines is cross magnetizing and distorts the main field and so weakens it and interferes with commutation. The flux produced by it is alternating and induces in the armature a back e.m.f. of armature inductance. Two methods of reducing this flux are used, both of which correspond to the use of interpoles on direct-current machines. A winding, called a compensating winding, is placed in slots in the pole faces as shown in Fig. 339. It is distributed over the

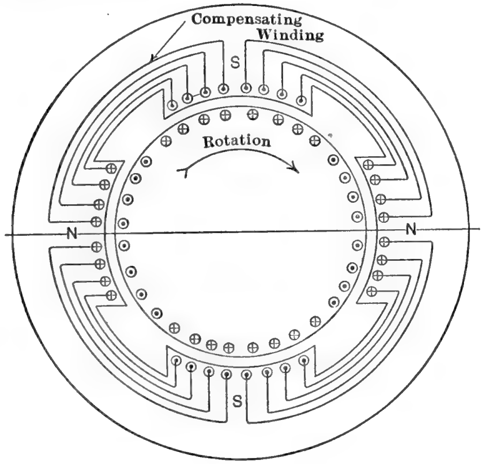


FIG. 339. Four-pole, single-phase, series motor with compensating winding.

whole periphery of the armature and exerts a m.m.f. opposing the armature m.m.f. and so limiting the cross flux to a very small value and reducing the armature inductance and reactance in the same proportion.

The m.m.f. of the compensating winding can be produced in two ways illustrated in Fig. 340 and Fig. 341. The first is called inductive compensation and the second conductive compensation.

(1) In the inductively compensated series motor the compensating m.m.f. is produced by short circuiting the compensating coil.

It then acts as the closed secondary of a transformer of which the armature is the primary. The m.m.f. of the compensating winding is almost equal to the m.m.f. of the armature but can never be greater than it and, therefore, overcompensation is not possible.

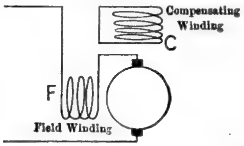


FIG. 340. Inductively compensated series motor.

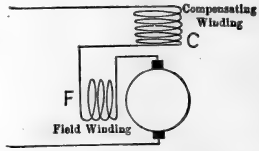


FIG. 341. Conductively compensated series motor.

The combined reactance of the armature and compensating winding corresponds to the reactance of a transformer on short circuit. The mutual flux is almost destroyed but the leakage fluxes remain.

(2) In the conductively compensated motor the compensating coil is connected in series with the field and armature and the amount of compensation can be varied. When the m.m.fs. of the two windings are equal there is no mutual flux and the combined reactance is a minimum. When the m.m.f. of the compensating winding is stronger than that of the armature the armature reaction flux is reversed but the reactance of the compensating winding is increased and so part of the advantage is lost, but the flux due to overcompensation assists commutation of the load current in the same way that interpoles do and is thus a great advantage.

A conductively compensated motor can be operated on direct current but an inductively compensated motor cannot since the compensating winding would not be effective and sparking would occur.

230. Commutation. Satisfactory commutation is very much more difficult to obtain in the alternating-current series motor than in the direct-current motor because, as may be seen in Fig. 342, the short-circuited coil is in the position of the short-circuited secondary of a transformer with the main field as primary and tends to have as many ampere turns induced in it as there are on a pair of field poles. This large short-circuit current interferes with commutation and must be reduced as far as possible. For this purpose high-resistance leads, called preventive leads, are connected between the coils and the commutator bars, as shown in Fig. 343, and narrow

carbon brushes of high contact resistance are used. The short-circuit current must pass through two resistance leads in series and is thus greatly reduced while the load current is carried by two or more in multiple. The resistance of one of the leads must be very much higher than that of an armature coil in order to reduce the current sufficiently.

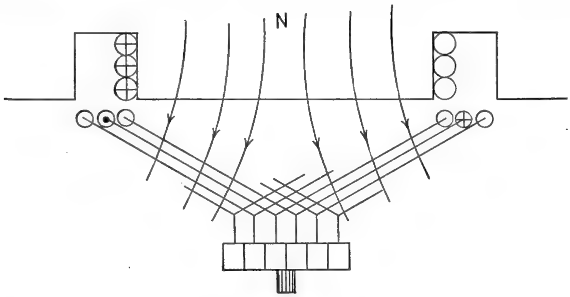


Fig. 342.

There are losses in the leads due to the resultant of the two currents flowing in them, but by increasing the resistance up to a certain point the short-circuit current is reduced and the combined loss is reduced.

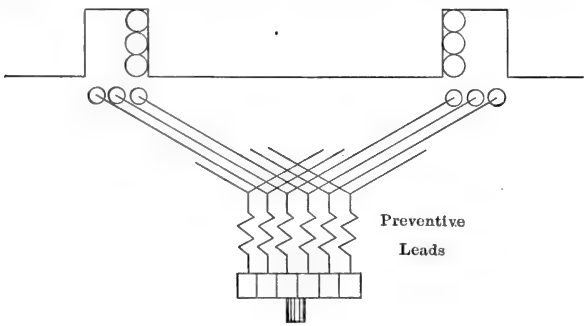


Fig. 343.

The resistance leads are not made of large enough capacity to carry the current continuously but under running conditions any one lead would only be in circuit for a short time. If the motor is stalled with power on the leads are likely to be destroyed.

The torque of the motor is very much improved by the use of

resistance leads since without them the large short-circuit current would weaken the main field and decrease the torque.

Fig. 344 shows the characteristic curves of a 150-horse-power, single-phase series motor. The torque and speed curves are very much the same shape as those of the direct-current series motor.

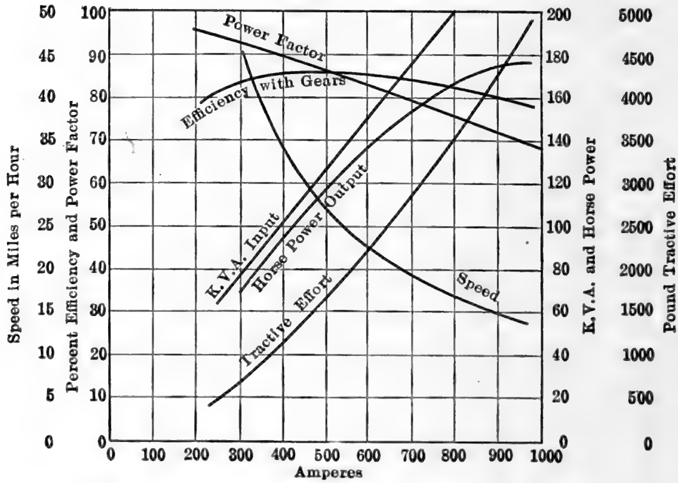


FIG. 344. Characteristic curves of a 25-cycle, 250-volt, 150-horse-power, single-phase series motor.

The power factor approaches unity at light load when the speed is high as explained above, but at full load it is still very good, reaching 90 per cent in some cases. At start and at low speeds it is low because the reactance of the motor is constant.

Efficiencies up to 85 per cent can be obtained but the motors must be designed more liberally than the corresponding direct-current motors and are therefore heavier and more expensive.

On account of unsatisfactory commutation alternating-current series motors are only built for voltages of 250 volts and under.

231. Repulsion Motor. In construction the repulsion motor resembles the single-phase series motor with conductive compensation. The armature is not connected in series with the field but is short circuited and receives its current by induction. (Figs. 345 to 347.)

The principal of its operation can be understood by reference to Figs. 345 to 347. In Fig. 345 the armature is shown short

circuited with its brushes in line with the field poles. Current is induced in it as in the secondary of a transformer and is very large but the torque exerted in both directions is the same and thus the resultant torque is zero. In Fig. 346, with the brushes

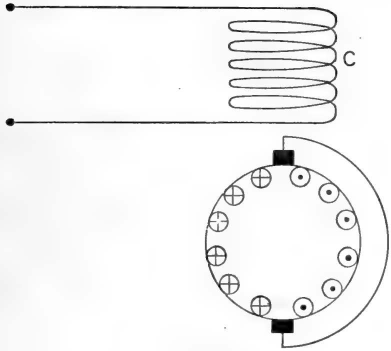


Fig. 345.

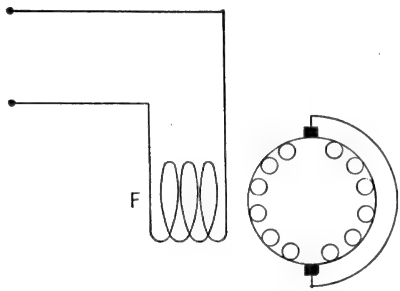


Fig. 346.

turned through 90 degrees there is no current induced in the armature and therefore no torque.

In order that the motor may exert torque the brushes must be placed in some intermediate position. The same result is accomplished by placing a second winding at right angles to the main field winding. This is shown as the compensating coil C in Fig. 347 and is carried in slots in the pole faces as in the series motor. The brushes are placed in line with this coil and the armature receives its current by induction from it. Torque is produced which is proportional to the product of the armature current induced by the compensating coil and the flux produced by the main field, but it is necessary to show that the current and flux are in time phase with one another.

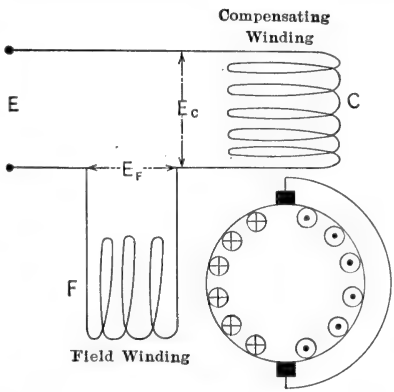


Fig. 347. Repulsion motor.

If voltage is impressed on the motor at rest, current flows in

both coils C and F. There is a large drop of voltage across F since its reactance is high, but only a very small drop across C since its reactance is low due to the presence of the short-circuited armature winding and thus at standstill a large flux passes through F and a small flux through C.

The flux in F is in time phase with the field current; the current in the armature is in phase opposition to the field current and therefore reaches its maximum at the same instant as the flux in F, and the torque which is proportional to their product retains its sign as they reverse together.

When the armature rotates an e.m.f. is generated between the brushes by the armature conductors cutting the flux from F. This e.m.f. is at every instant proportional to the product of the flux and the speed and is in phase with the flux and is therefore 90 degrees behind the e.m.f. across F. The armature now acts at the primary of a transformer with the compensating coil as secondary and it produces a flux which transfers the speed e.m.f. to the compensating coil and the coil C therefore consumes a large component of the impressed e.m.f.

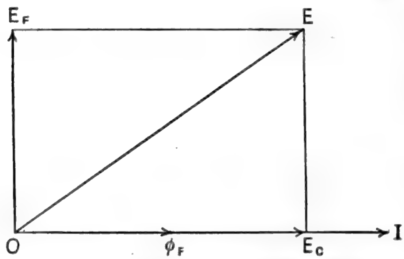


FIG. 348. E.m.f. and current in a repulsion motor.

In Fig. 348 I is the line current which flows in the coils F and C.

Φ_F is the flux in F, which is in phase with the current.

E_F is the component of impressed e.m.f. across the terminals of F.

E_S is the e.m.f. generated in the armature by rotation and transferred to the compensating coil. (Shown as E_C.)

E_C is the component of impressed e.m.f. across the terminals of C; it is equal to E_S if the coil C has the same number of turns as the armature, and it is in phase with it.

$E = \sqrt{E_F^2 + E_C^2}$ is the constant line voltage impressed on the motor.

The e.m.f. consumed by the impedance of the armature and compensating winding is neglected. As the speed increases the e.m.f. E_C increases and E_F decreases, the flux in the main field F decreases and the current and torque decrease.

At start, when the drop across C is small, the current is large and the main field F is very strong. The repulsion therefore gives a good starting torque. The field at start will be decreased to a certain extent by the current in the short-circuited coil undergoing commutation, as in the series motor.

232. Commutation. In the single-phase series motor and the repulsion motor there are two currents to be commutated, (1) the load current and (2) the short-circuit current produced in the coil under the brush by the alternating flux of the main field.

(1) To reverse the load current a m.m.f. is required opposing the m.m.f. of armature reaction and strong enough to produce a flux in the opposite direction to the armature reaction flux. Such a flux can be produced by interpoles placed between the main poles and excited by a winding in series with the main field or it can be produced by a compensating winding. The conductively compensating winding is the only one which can give perfect commutation since its m.m.f. can be made stronger than the armature m.m.f. Commutation is assisted by the use of high-resistance carbon brushes.

(2) To eliminate the short-circuit current in the coil under the brush an e.m.f. must be generated in the coil equal and opposite to the e.m.f. producing the short-circuit current. The neutralizing e.m.f. cannot be generated by the alternation of a magnetic flux through the coil, since that would require a flux equal and opposite to the field flux and would destroy the field of the motor. The required e.m.f. can, however, be generated by the rotation of the armature through a commutating field of the proper intensity and position, but the field must be in quadrature with the main field in both time and space. In the repulsion motor under running conditions such a field is produced in the compensating coil C. The intensity of the field varies with the speed of the motor. Near synchronous speed the e.m.f. is entirely neutralized and the current is wiped out. Below synchronous speed the current is reduced and above synchronous speed another current is produced as objectionable as before and commutation becomes bad again. At standstill no neutralizing e.m.f. is produced.

In the single-phase series motor there is no field in quadrature with the main field in time and so no neutralizing e.m.f. can be produced but the short-circuit current is reduced by using high-resistance preventive leads as explained in Art. 230.

Thus near synchronous speed the commutation of the repulsion motor is better than that of the series motor.

Since preventive leads are not used in the repulsion motor the short-circuit current in it at start will be greater than in the series motor and will weaken the main field and decrease the starting torque.

While running the short-circuit current is not so great, since it cannot reach its maximum value on account of the self-inductance of the coil.

The repulsion motor cannot be operated more than 40 per cent above synchronous speed on account of commutation troubles.

A very large number of alternating-current commutator motors, differing in certain details from the two described here, have been designed and are in successful operation but the main principles are the same.

CHAPTER IX

CONVERTERS

233. Rotary Converter. The rotary converter or synchronous converter is a combination of a synchronous motor and a direct-current generator. It receives alternating current and converts it into direct current.

The armature winding is an ordinary direct-current winding and may be either series or multiple. It is connected to a commutator and taps are taken out from it at equidistant points and connected to slip rings. The alternating current is delivered to the slip rings either single-phase, two-phase, three-phase or six-phase and drives the armature as a synchronous motor. The same armature conductors generate and carry the direct current.

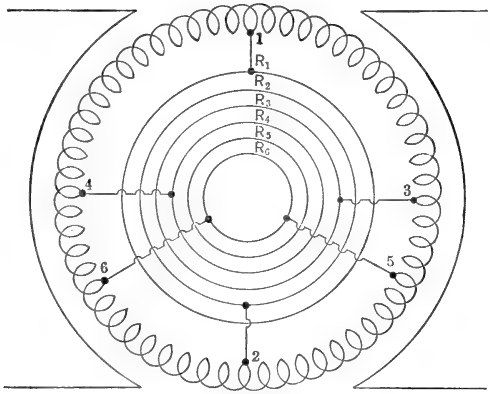


FIG. 349. Ring-wound rotary converter.

Fig. 349 shows a ring-wound bipolar armature tapped for single-, two-, or three-phase currents; single-phase 1 to 2 or 3 to 4; two-phase 1 to 2 and 3 to 4; three-phase 1 to 5, 5 to 6 and 6 to 1.

Fig. 350 shows a six-circuit multiple winding for a six-pole, three-phase rotary converter and Fig. 351 shows a two-circuit or series winding for an eight-pole, three-phase converter.

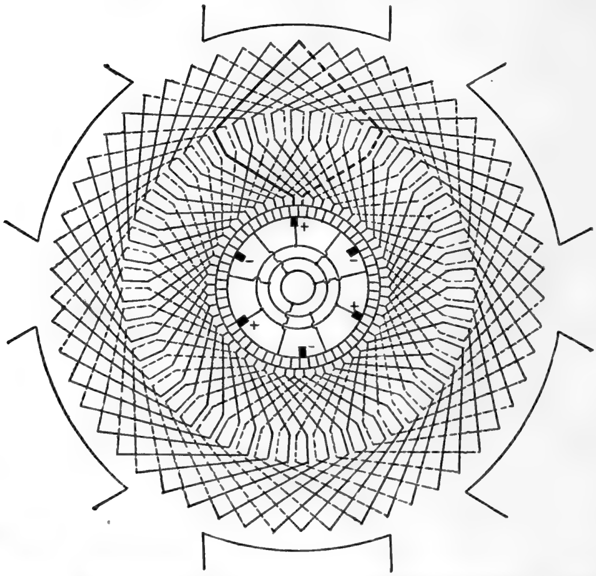


FIG. 350. Multiple-drum winding for a three-phase rotary converter.

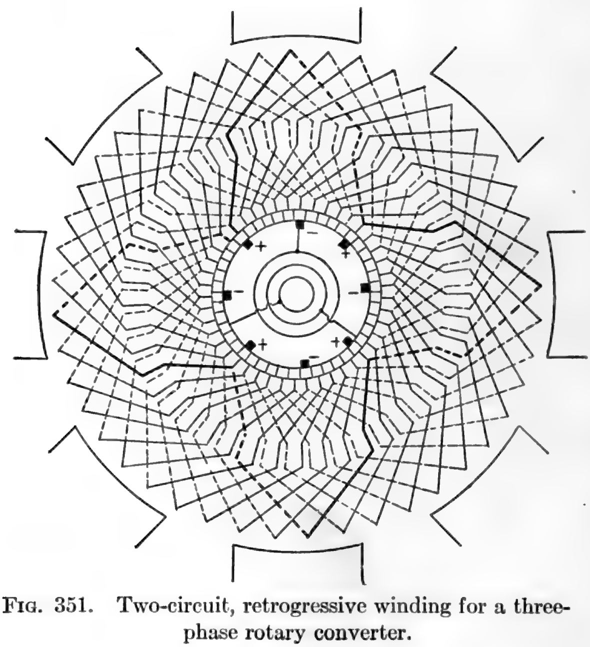


FIG. 351. Two-circuit, retrogressive winding for a three-phase rotary converter.

In a series-wound armature the total number of coils must be divisible by the number of phases and in a multiple-wound armature the number of coils per pair of poles must be divisible by the number of phases.

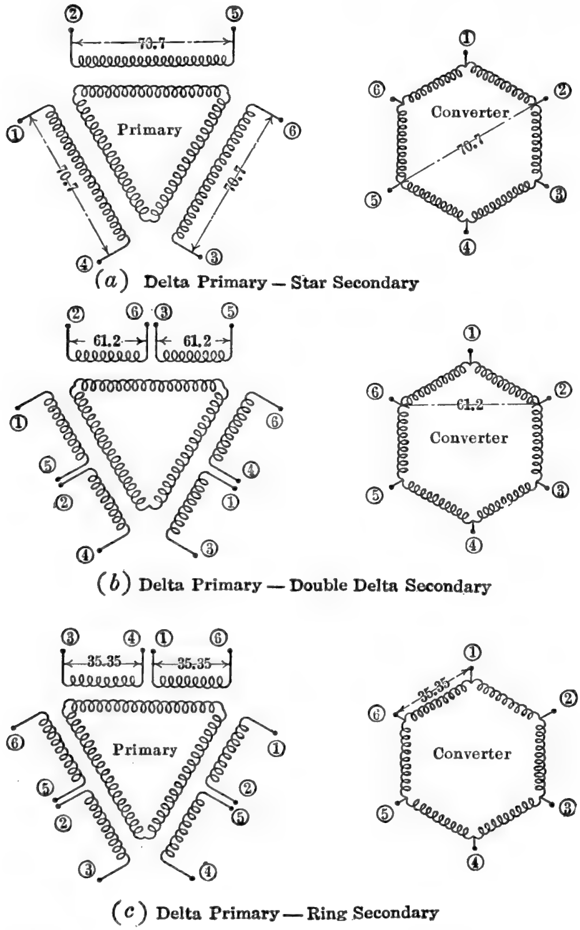


FIG. 352. Six-phase power from three-phase circuits.

Six-phase converters are operated from three-phase circuits. Three methods of connecting the supply transformers are shown in Fig. 352.

234. Field Excitation. The fields of a rotary converter are excited by direct current, usually supplied by an exciter mounted

on the same shaft. For overcompounding, a series field winding is added. Its action is explained in Art. 242.

235. Ratios of E.M.Fs. and Currents. With the brushes fixed on the no-load neutral line, the direct and alternating e.m.f.'s generated in the converter bear a definite relation to each other and one cannot be varied without varying the other. At unity power factor the alternating and direct currents in the armature also bear a definite relation to each other if the current required to supply the converter losses is neglected.

Since the alternating and direct currents flow in the same armature conductors and in opposite directions, the e.m.f. consumed in the armature is small and the power loss is small. In the following analysis these quantities will be neglected and the alternating current will be assumed to be in phase with the impressed e.m.f. This condition can be obtained in practice by properly adjusting the exciting current.

Take first the case of the single-phase converter, Fig. 353.

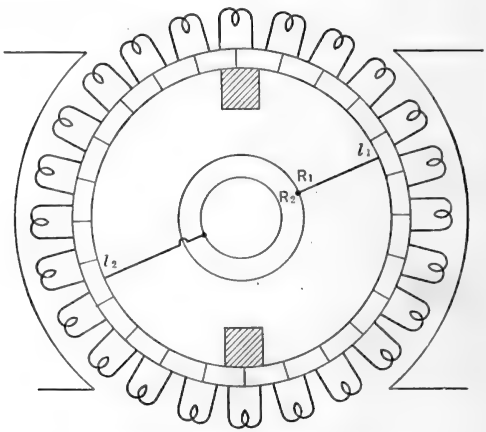


FIG. 353. Single-phase converter.

Let E = direct voltage of the converter.

I = direct-current output.

E_1 = effective value of the alternating supply voltage, whether single phase or polyphase.

I_1 = alternating current in the supply lines.

I' = alternating current in the armature winding.

The voltage between the leads l_1 and l_2 or between the slip rings R_1 and R_2 is alternating and reaches its maximum value when l_1 and l_2 are under the brushes and it is then equal to the direct voltage of the machine. Therefore, in a single-phase converter the direct voltage is equal to the maximum value of the alternating voltage and thus

$$E = \sqrt{2}\,E_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (339)$$

or

$$E_1 = \frac{E}{\sqrt{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (340)$$

Neglecting losses and phase displacements the output of the converter is equal to the volt-amperes input or[7]

$$EI = E_1 I_1$$

and the alternating current in the line is

$$I_1 = \frac{EI}{E_1} = \frac{EI}{\frac{E}{\sqrt{2}}} = \sqrt{2}\,I. \quad . \quad . \quad . \quad . \quad . \quad (341)$$

The alternating current in the winding is

$$I' = \frac{I_1}{p},$$

where p is the number of circuits in multiple through the winding. In the bipolar machine in Fig. 353, $p = 2$ and, therefore,

$$I' = \frac{I_1}{2} = \frac{\sqrt{2}\,I}{2} = \frac{I}{\sqrt{2}}. \quad . \quad . \quad . \quad . \quad . \quad (342)$$

236. Two-phase or Quarter-phase Converter. When four collector rings R_1, R_2, R_3 and R_4, Fig. 354, are connected to four equidistant points l_1, l_2, l_3 and l_4, the machine is a two-phase or quarter-phase converter. The two voltages R_1 to R_2 and R_3 to R_4 are equal and are in quadrature, forming a two-phase system; the four voltages R_1 to R_3, R_3 to R_2, R_2 to R_4 and R_4 to R_1 are all equal and form a four-phase or quarter-phase system.

The voltage between lines or the voltage per phase of the two-phase supply is

$$E_1 = \frac{E}{\sqrt{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (343)$$

The voltage between adjacent rings or the quarter-phase voltage is

$$E_1' = \frac{E_1}{\sqrt{2}} = \frac{E}{2\sqrt{2}}. \qquad (344)$$

Assuming the volt-amperes input two-phase to be equal to the direct-current output, that is,

$$2 E_1 I_1 = EI,$$

the alternating current per line is

$$I_1 = \frac{EI}{2E_1} = \frac{EI}{2\frac{E}{\sqrt{2}}} = \frac{I}{\sqrt{2}}. \qquad (345)$$

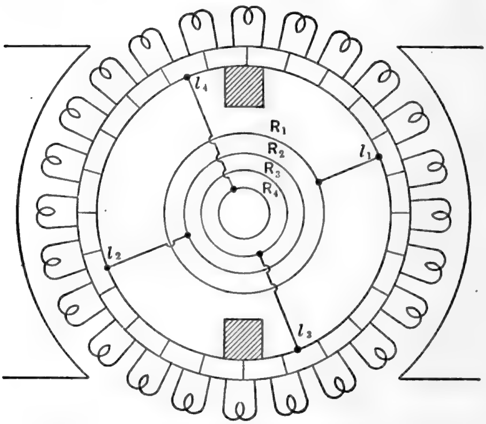


FIG. 354. Two-phase or quarter-phase converter.

The alternating current in the winding is the resultant of two currents $\frac{I_1}{2} = \frac{I}{2\sqrt{2}}$ in quadrature and its value is therefore

$$I' = \sqrt{2}\frac{I}{2\sqrt{2}} = \frac{I}{2}. \qquad (346)$$

237. Three-phase Converter. With three collector rings R_1, R_2 and R_3, Fig. 355, connected to three equidistant points l_1, l_2 and l_3 the machine is a three-phase converter.

The e.m.f. between each of the rings and the neutral point or the "star" e.m.f. is equal to half of the single-phase voltage. It is shown as E_S in Fig. 356.

$$E_S = \frac{E}{2\sqrt{2}}. \qquad (347)$$

The e.m.f. between rings or "delta" e.m.f. is

$$E_1 = \sqrt{3}\, E_S = \frac{\sqrt{3}\, E}{2\sqrt{2}} = 0.612\, E. \quad . \quad . \quad . \quad (348)$$

The power input is

$$\sqrt{3}\, E_1 I_1 = 3\, E_S I_1 = 3\, E_1 I'$$

and is equal to the output EI.

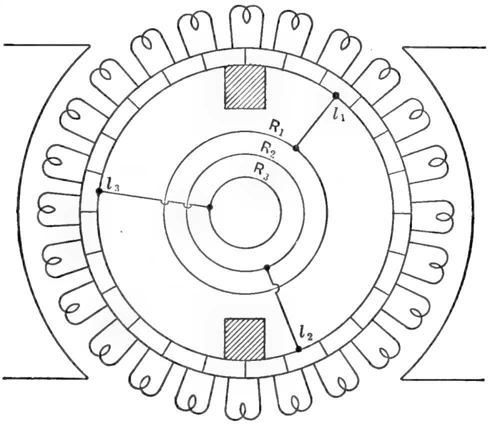


FIG. 355. Two-phase converter.

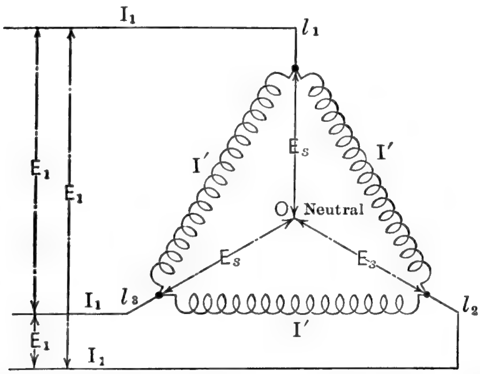


FIG. 356. E.m.fs. and currents in a three-phase converter.

Thus the line current is

$$I_1 = \frac{EI}{\sqrt{3}\, E_1} = \frac{EI}{\sqrt{3} \cdot \dfrac{\sqrt{3}\, E}{2\sqrt{2}}} = \frac{2\sqrt{2}}{3} I = 0.943\, I, \quad (349)$$

and the alternating current in the winding is

$$I' = \frac{I_1}{\sqrt{3}} = \frac{2\sqrt{2}}{3\sqrt{3}} I = 0.545\,I. \quad . \quad . \quad . \quad . \quad (350)$$

238. n-phase Converter. For an n-phase converter, Fig. 357, the winding must be tapped at n equidistant points. The e.m.f. between each of the rings and the neutral point or the "star" e.m.f. is as before

$$E_S = \frac{E}{2\sqrt{2}}.$$

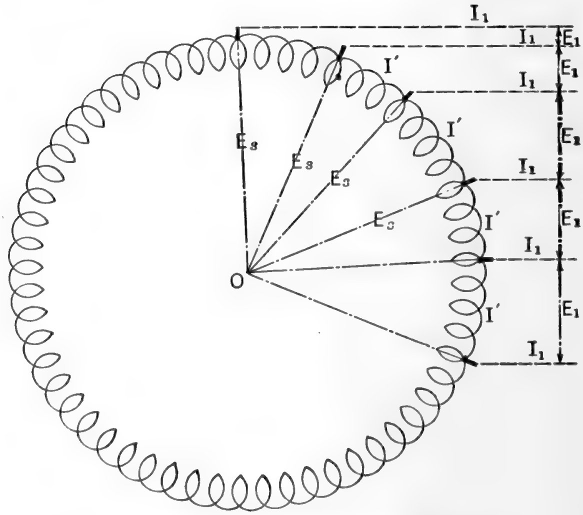


FIG. 357. n-phase converter.

The e.m.f. between rings or the e.m.f. between lines is the vector difference between two e.m.fs. E_S displaced at $\frac{2\pi}{n}$ radians. (Fig. 358.) Thus

$$E_1 = 2\,E_S \sin\frac{\pi}{n} = \frac{E \sin\frac{\pi}{n}}{\sqrt{2}}. \quad . \quad . \quad . \quad . \quad (351)$$

The power input is

$$nE_SI_1 = nE_1I'$$

and is equal to the output EI.

Therefore the alternating current in the line is

$$I_1 = \frac{EI}{nE_S} = \frac{EI}{n\frac{E}{2\sqrt{2}}} = \frac{2\sqrt{2}}{n}I, \quad . \quad . \quad . \quad . \quad (352)$$

and the alternating current in the winding is

$$I' = \frac{EI}{nE_1} = \frac{EI}{n\frac{E\sin\frac{\pi}{n}}{\sqrt{2}}} = \frac{\sqrt{2}\,I}{n\sin\frac{\pi}{n}}. \quad . \quad . \quad . \quad . \quad (353)$$

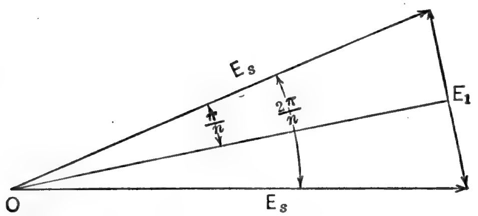


FIG. 358.

The values obtained above for the e.m.fs. and currents in single-phase, two-phase and three-phase converters can also be obtained by substituting the proper values of n in equations 351, 352 and 353; single-phase $n = 2$, two-phase or four-phase $n = 4$, and three-phase $n = 3$. The results are tabulated in Fig. 359.

Type	Single-phase $n=2$	Three-phase $n=3$	Two-phase or four-phase $n=4$	Six-phase $n=6$	n-phase
E.m.f. between collector rings or line e.m.f. E_1..........	$\frac{E}{\sqrt{2}}$	$\frac{\sqrt{3}\,E}{2\sqrt{2}}$	$\frac{E}{2}$	$\frac{E}{2\sqrt{2}}$	$\frac{E\sin\frac{\pi}{n}}{\sqrt{2}}$
Current per line I_1	$\sqrt{2}\,I$	$\frac{2\sqrt{2}\,I}{3}$	$\frac{I}{\sqrt{2}}$	$\frac{\sqrt{2}\,I}{3}$	$\frac{2\sqrt{2}\,I}{n}$
Current in the winding I'..............	$\frac{I}{\sqrt{2}}$	$\frac{2\sqrt{2}\,I}{3\sqrt{3}}$	$\frac{I}{2}$	$\frac{\sqrt{2}\,I}{3}$	$\frac{\sqrt{2}\,I}{n\sin\frac{\pi}{n}}$

FIG. 359.

These ratios of currents only hold on the assumption that the power factor is unity and that the efficiency is 100 per cent.

The power factor can be maintained approximately unity by adjusting the excitation, but the power losses cannot be eliminated and the values of alternating current in the table must be increased by the small component required to supply the losses in the machine. When the power factor is not unity the reactive or wattless components of current must be added to the power components.

The ratios of e.m.fs. are the ratios of the generated e.m.fs. and can only approximately represent the ratios of terminal e.m.fs. since components of e.m.f. are consumed in the resistance and reactance of the armature. The ratios also depend on the assumption that the alternating e.m.f. wave is a sine wave. If the wave is peaked, the ratio of the effective value to the maximum value is less than $\frac{1}{\sqrt{2}}$ and the values of the alternating e.m.fs. must be reduced. If the wave is flat topped the ratio of effective to maximum value is greater than $\frac{1}{\sqrt{2}}$ and the values of the e.m.fs. must be increased.

239. Wave Forms of Currents in the Armature Coils. The current in any armature coil is the difference between the alternating-current input and the direct-current output.

In Fig. 360, l_1 and l_2 are the two leads of one of the n-phases of a converter, a is the coil next to one lead and c is the coil in the centre of the phase. The alternating e.m.f. and the power component of the alternating current in the phase l_1 to l_2 are maximum when this section of the winding is midway between the brushes and they are both zero when the centre coil c is under the brush.

Fig. 361 shows the resultant of the alternating and direct currents in coil c during one revolution. The alternating current is opposed to the direct current and is zero when the direct current reverses as the coil passes under the brush. The current in the centre coil is, therefore, less than the current in any other coil in the phase when the power factor is unity.

Fig. 362 shows the current in coil a next to one of the leads. The alternating and direct currents are not directly opposing and the resultant current is greater than in the centre coil c.

The coils next to the leads, therefore, carry larger currents than the coils in the centre of the phases and they rise to a higher temperature.

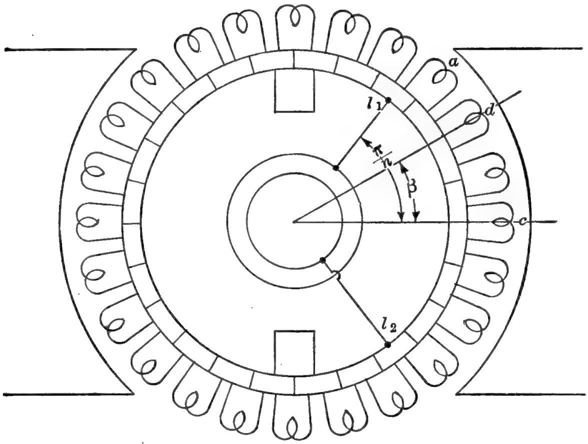


FIG. 360.

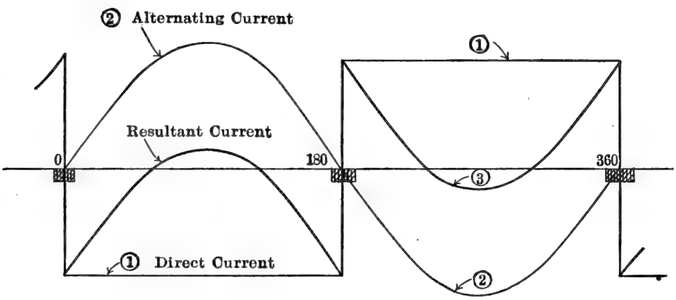


FIG. 361. Current in coil c, Fig. 360, at unity power factor.

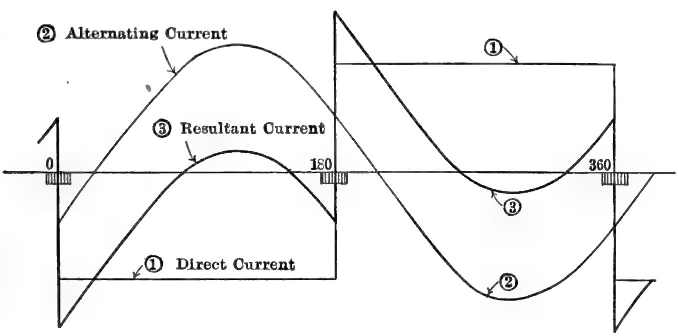


FIG. 362. Current in coil a, Fig. 360, at unity power factor.

The worst condition of local heating occurs in the coil next the lead of a single-phase converter, Fig. 363. The alternating current has its maximum value when the direct current reverses.

When the power factor is not unity the minimum resultant current will not occur in the centre coil of a phase but in a coil displaced from it by the angle of lag or lead of the current.

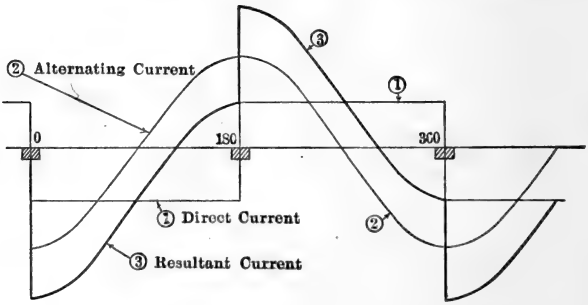


FIG. 363. Current in the coil next to the lead of a single-phase converter at unity power factor.

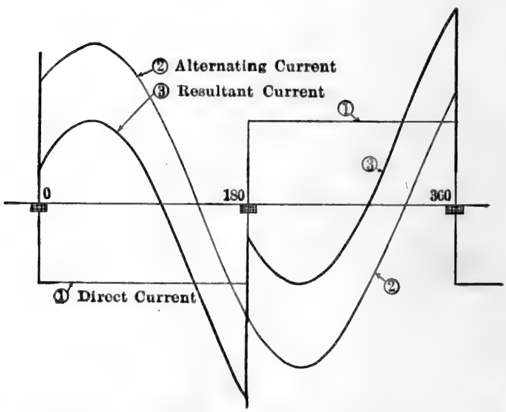


FIG. 364. Current in coil c, Fig. 360, at 70 per cent power factor.

Fig. 364 shows the current in coil c when the power factor is 70 per cent and a component of lagging current equal to the power current flows in the armature.

240. Heating Due to Armature Copper Loss. Take the case of a two-pole, n-phase armature, Fig. 360, and use the same notation as before. In the centre coil c of the phase, the direct

current is $\frac{I}{2}$ and the effective value of the alternating current is

$$I' = \frac{\sqrt{2}\, I}{n \sin \frac{\pi}{n}}.$$

The instantaneous value of the alternating current is

$$i = \sqrt{2}\, I \sin \theta = \frac{2\, I}{n \sin \frac{\pi}{n}} \sin \theta, \quad . \quad . \quad . \quad . \quad . \qquad (354)$$

and the instantaneous value of the resultant current is

$$i_0 = \frac{2\, I}{n \sin \frac{\pi}{n}} \sin \theta - \frac{I}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (355)$$

In an armature coil d displaced by angle β from the centre of the phase the alternating current is

$$i = \sqrt{2}\, I' \sin(\theta - \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (356)$$

and the instantaneous value of the resultant current is

$$i_\beta = \frac{2\, I}{n \sin \frac{\pi}{n}} \sin (\theta - \beta) - \frac{I}{2}$$

$$= \frac{I}{2} \left\{ \frac{4 \sin (\theta - \beta)}{n \sin \frac{\pi}{n}} - 1 \right\}. \quad . \quad . \quad . \quad . \quad . \qquad (357)$$

The effective value of the resultant current is

$$I_\beta = \sqrt{\frac{1}{\pi} \int_0^\pi i_\beta^2 \, d\theta} = \frac{I}{2} \sqrt{\frac{1}{\pi} \int_0^\pi \left\{ \frac{4 \sin (\theta - \beta)}{n \sin \frac{\pi}{n}} - 1 \right\}^2 d\theta} \quad . \quad . \qquad (358)$$

$$= \frac{I}{2} \sqrt{\frac{1}{\pi} \int_0^\pi \left\{ \frac{16 \sin^2(\theta - \beta)}{n^2 \sin^2 \frac{\pi}{n}} - \frac{8 \sin(\theta - \beta)}{n \sin \frac{\pi}{n}} + 1 \right\} d\theta}$$

$$= \frac{I}{2} \sqrt{\frac{1}{\pi} \int_0^\pi \left[\frac{8\{1 - \cos 2\, (\theta - \beta)\}}{n^2 \sin^2 \frac{\pi}{n}} - \frac{8 \sin(\theta - \beta)}{n \sin \frac{\pi}{n}} + 1 \right] d\theta}$$

$$= \frac{I}{2} \sqrt{\frac{1}{\pi} \left[\frac{8}{n^2 \sin^2 \frac{\pi}{n}} \left\{ \theta - \frac{\sin 2\, (\theta - \beta)}{2} \right\} + \frac{8 \cos (\theta - \beta)}{n \sin \frac{\pi}{n}} + \theta \right]_0^\pi}$$

$$= \frac{I}{2}\sqrt{\frac{1}{\pi}\left[\frac{8\pi}{n^2\sin^2\frac{\pi}{n}} - \frac{16\cos\beta}{n\sin\frac{\pi}{n}} + \pi\right]}$$

$$= \frac{I}{2}\sqrt{\frac{8}{n^2\sin^2\frac{\pi}{n}} - \frac{16\cos\beta}{n\pi\sin\frac{\pi}{n}} + 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (359)$$

Since $\frac{I}{2}$ is the current in the coil when the machine is operating as a direct-current generator, the ratio of the power lost in the coil when operating as a converter to that lost when operating as a direct-current generator with the same output is

$$h_\beta = \left[\frac{I_\beta}{\frac{I}{2}}\right]^2 = \frac{8}{n^2\sin^2\frac{\pi}{n}} - \frac{16\cos\beta}{n\pi\sin\frac{\pi}{n}} + 1 \quad . \quad . \quad . \quad (360)$$

and this is the ratio of the coil heating under the two conditions.

This ratio is a maximum for the coil next to the alternating leads l_1 or l_2, where $\beta = \frac{\pi}{n}$, and it is

$$h_{\max.} = \frac{8}{n^2\sin^2\frac{\pi}{n}} - \frac{16\cos\frac{\pi}{n}}{n\pi\sin\frac{\pi}{n}} + 1. \quad . \quad . \quad . \quad . \quad (361)$$

It is a minimum for the centre coil of the phase, where $\beta = 0$, and is

$$h_0 = \frac{8}{n^2\sin^2\frac{\pi}{n}} - \frac{16}{n\pi\sin\frac{\pi}{n}} + 1. \quad . \quad . \quad . \quad (362)$$

The ratio of the total power lost in the armature of the converter to that lost when the machine is operating as a direct-current generator with the same output is found by integrating the ratio h_β over one half phase from $\beta = \frac{\pi}{n}$ to $\beta = 0$ and taking the average. It is

$$H = \frac{n}{\pi}\int^{\frac{\pi}{n}} h_\beta\, d\beta = \frac{n}{\pi}\int_0 \left(\frac{8}{n^2\sin^2\frac{\pi}{n}} - \frac{16\cos\beta}{n\pi\sin\frac{\pi}{n}} + 1\right) d\beta$$

$$= \frac{n}{\pi}\left[\frac{8\,\beta}{n^2 \sin^2 \frac{\pi}{n}} - \frac{16 \sin \beta}{n\pi \sin \frac{\pi}{n}} + \beta\right]_0^{\frac{\pi}{n}}$$

$$= \frac{8}{n^2 \sin^2 \frac{\pi}{n}} - \frac{16}{\pi^2} + 1, \qquad (363)$$

and this is the relative armature heating under the two conditions.

To get the same loss in the armature of a converter and the same heating as in the direct-current generator, the armature current and the output may be increased in the ratio $\frac{1}{\sqrt{H}}$.

The values of H and $\frac{1}{\sqrt{H}}$ for the various polyphase converters are tabulated in Fig. 365.

Type	Direct-current generator	Single-phase $n=2$	Three-phase $n=3$	Two-phase or four-phase $n=4$	Six-phase $n=6$
Relative armature heating H..........	1.00	1.37	0.55	0.37	0.26
Rating by armature heating $\frac{1}{\sqrt{H}}$........	1.00	0.85	1.34	1.64	1.96

FIG. 365.

For a single-phase converter, $n = 2$, the value of $\frac{1}{\sqrt{H}}$ is 0.85, and, therefore, the output of a machine as a single-phase converter is only 85 per cent of its output as a direct-current generator for the same temperature rise.

For a three-phase converter $\frac{1}{\sqrt{H}} = 1.34$ and therefore the output is 34 per cent greater than as a direct-current generator. For a six-phase converter the output is increased 96 per cent.

These values only hold if the alternating current is in phase with the impressed e.m.f. When leading or lagging currents flow in the armature the heating is very largely increased and the rating must be decreased. The rating of a machine as a three-phase

converter will be the same as when operated as a direct-current generator when the power factor is about 85 per cent and the rating of a machine as a six-phase converter will be the same as when operated as a direct-current generator when the power factor is about 75 per cent.

Rotary converters should, therefore, be operated at unity power factor at full load and overload.

241. Armature Reaction. The armature reaction of a rotary converter is the resultant of the armature reactions of the machine as a direct-current generator and as a synchronous motor. The direct-current brushes are usually placed at right angles to the field poles and therefore the direct-current exerts a m.m.f. in quadrature behind the field m.m.f., Fig. 366. The power com-

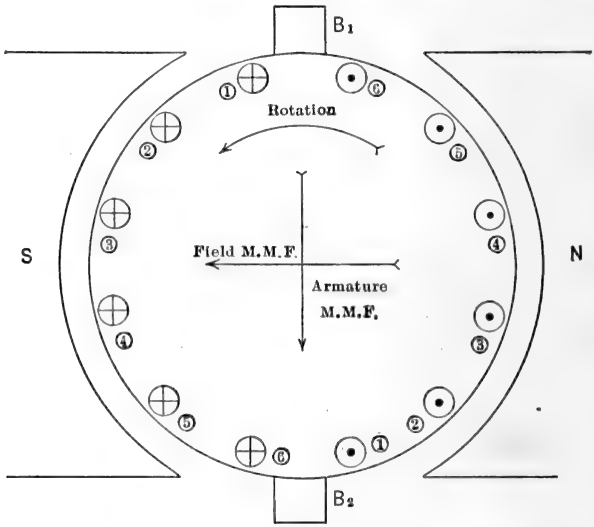


FIG. 366. Direction of m.m.fs. in a direct-current generator.

ponent of the armature current in a synchronous motor exerts a m.m.f. in quadrature ahead of the field m.m.f. and it is therefore opposed to the m.m.f. of the direct current.

If Z is the number of conductors on the armature of a bipolar generator and $\frac{I}{2}$ is the direct current in each conductor, the armature m.m.f. is the resultant of $\frac{Z}{2}$ m.m.fs. of magnitude $\frac{I}{2}$ uni-

formly distributed over the circumference of the armature and it is therefore less than that of a concentrated winding in the ratio $\frac{2}{\pi}$. (Fig. 367.) The m.m.f of the direct current in the armature of a converter is thus

$$M_d = \frac{Z}{2} \cdot \frac{I}{2} \cdot \frac{2}{\pi} = \frac{ZI}{2\pi} \quad \ldots \ldots \quad (364)$$

in quadrature behind the field m.m.f.

If the machine is connected as an n-phase converter, the number of turns per phase is $\frac{Z}{2n}$ and the effective value of the alternating current in each is

$$I' = \frac{\sqrt{2}\, I}{n \sin \frac{\pi}{n}},$$

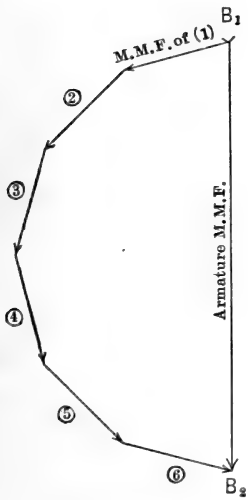


FIG. 367.

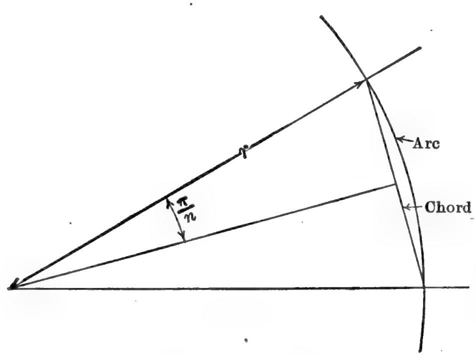


FIG. 368.

and the m.m.f. per phase in effective ampere turns is

$$m' = \frac{ZI'}{2n} = \frac{ZI}{\sqrt{2}\, n^2 \sin \frac{\pi}{n}}. \quad \ldots \ldots \quad (365)$$

These ampere turns are distributed over $\frac{1}{n}$th of the circumference of the armature and their resultant is, Fig. 368,

$$m = \frac{ZI'}{2\,n} \cdot \frac{\text{chord}}{\text{arc}} = \frac{ZI'}{2\,n} \cdot \frac{2\,r \sin \frac{\pi}{n}}{\frac{2\,\pi r}{n}}$$

$$= \frac{ZI}{\sqrt{2}\,n^2 \sin \frac{\pi}{n}} \cdot \frac{2\,r \sin \frac{\pi}{n}}{\frac{2\,\pi r}{n}}$$

$$= \frac{ZI}{\sqrt{2}\,\pi n}. \qquad (366)$$

The maximum value of the m.m.f. per phase is

$$m_0 = \sqrt{2}\,m = \frac{ZI}{\pi n} \text{ ampere turns.} \qquad (367)$$

To find the resultant m.m.f. of the alternating current in the armature it is necessary to combine n m.m.fs. of maximum value $m_0 = \frac{ZI}{\pi n}$ displaced in direction by angle $\frac{2\pi}{n}$ and displaced in phase by $\frac{1}{n}$th of a period or by angle $\frac{2\,\pi}{n}$.

In Fig. 369 phase 1 is shown in the position of maximum m.m.f. if the power factor is unity. The direction of the m.m.f. is OB and it is in quadrature ahead of the field m.m.f. If time and angular displacement are measured from OB, then at time t and angle θ the m.m.f. of phase 1 is $m_0 \cos \theta$ in direction OB_1, and its component in direction OB is $m_0 \cos^2 \theta$.

At time t the m.m.f. of phase 2 is $m_0 \cos \left(\theta + \frac{2\,\pi}{n}\right)$ making an angle $\left(\theta + \frac{2\,\pi}{n}\right)$ with OB and its component in direction OB is $m_0 \cos^2 \left(\theta + \frac{2\,\pi}{n}\right)$.

The resultant m.m.f. of the n phases in the direction OB at any time t is

$$M_a = m_0 \left\{ \cos^2 \theta + \cos^2 \left(\theta + \frac{2\,\pi}{n}\right) + + + \cos^2 \left(\theta + \frac{2\,(n-1)\,\pi}{n}\right) \right\}$$

$$= m_0 \times n \times \text{average } (\cos)^2 = m_0 \frac{n}{2}$$

since the average $\cos^2$ is $= \frac{1}{2}$.

The resultant m.m.f. of the n phases in the direction at right angles to OB or in line with the field m.m.f. is zero.

Therefore the resultant m.m.f. of the alternating current in the converter armature has a constant value

$$M_a = m_0 \frac{n}{2} = \frac{ZI}{\pi n} \cdot \frac{n}{2} = \frac{ZI}{2\pi} \quad . \quad . \quad . \quad . \quad (368)$$

and is in quadrature ahead of the field m.m.f. It is thus equal to the m.m.f. of the direct current and is opposed to it and the resultant armature reaction of the direct current and of the corresponding power component of the alternating current is zero.

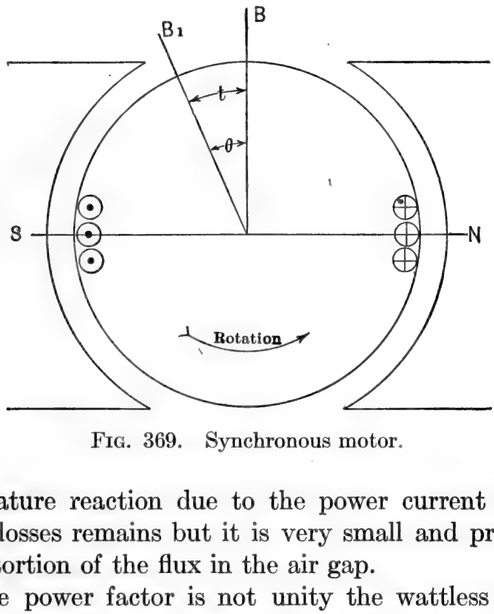


FIG. 369. Synchronous motor.

The armature reaction due to the power current required to supply the losses remains but it is very small and produces only a slight distortion of the flux in the air gap.

When the power factor is not unity the wattless currents in the armature exert m.m.fs., as in the synchronous motor, which act in line with the field m.m.f. and are either magnetizing or demagnetizing but are not distorting.

Thus in the rotary converter there is very little field distortion or very little shifting of the neutral points under load. As a result the limit of overload set by commutation is much higher than in the direct-current generator. This is very important in the case of converters supplying railway loads where the load factor is usually below 50 per cent. The overload capacities for

short periods must be high. In some cases when using interpoles momentary overloads of 200 per cent are permitted.

242. Compounding. When the field current of a rotary converter is varied wattless lagging or leading currents flow in the armature and magnetize or demagnetize the field and bring it back to its former strength, giving the same generated voltages as before, but there is no change in the direct voltage since it has a fixed relation with the impressed voltage. To vary the direct voltage the impressed alternating voltage must be varied. This can be done (1) by using variable-ratio supply transformers, (2) by connecting potential regulators in the lines or (3) by connecting reactance coils in the lines and drawing wattless lagging or leading currents through them by varying the field excitation. The first method has the disadvantage of a step-by-step regulation; the second requires expensive apparatus but gives very good regulation and can be made automatic; the third is inexpensive and may be made automatic by placing a series winding on the converter fields. The variation of the impressed voltage due to the reactance coils is explained as follows: The e.m.f. of self-inductance lags 90 degrees behind the current; thus, when the converter is under-excited and a component of current lagging 90 degrees behind the impressed e.m.f. flows through the reactance coil, the e.m.f. of self-inductance due to it lags 180 degrees behind the impressed e.m.f. and therefore opposes and lowers it. When the converter is over-excited and a component of current 90 degrees ahead of the impressed e.m.f. flows through the reactance coil, its e.m.f. of self-inductance is in phase with the impressed e.m.f. and raises it.

Therefore when reactance coils are connected in the supply lines an increase of the field excitation raises the impressed e.m.f. and so raises the direct voltage, and a decrease of the field excitation lowers the impressed e.m.f. and so lowers the direct voltage. The result is the same as in the direct-current generator but is produced in a different way.

If the converter is provided with a series winding the reactance coils cause the direct voltage to rise automatically with increase of load.

243. Starting. Converters may be started in various ways: (1) If reduced voltage is impressed on the slip rings the converter will start as a synchronous motor and come up to full speed as explained in Art. 169. Special supply transformers are used

with taps on the secondaries to give from one third to one half of full voltage at start. When synchronous speed is reached the field circuit is closed and full voltage is impressed. The load circuit must not be connected at start since below synchronous speed the voltage between the direct-current brushes is alternating at the frequency of slip.

By this method the trouble of synchronizing is eliminated and the machine can be put in operation very quickly, but a large current at low power factor is drawn from the lines.

(2) Converters may be started from the direct-current end if suitable power is available in the station, but a longer time is required to put them in operation than when starting from the alternating-current end and they must be synchronized.

(3) If an induction motor with a smaller number of poles than the converter and consequently a higher synchronous speed is mounted on the same shaft it may be used to start the converter. This method of starting requires synchronizing and thus takes longer than (1) and the induction motor draws a large lagging current. It has, therefore, the disadvantages of the two first methods and in addition requires the extra starting motor.

244. Frequencies and Voltages. Converters are built for both 25 and 60 cycles, but for large outputs the lower frequency is more satisfactory since on account of the larger number of poles required the commutation in a 60-cycle converter is not so good as in a 25-cycle machine.

Any direct voltage up to 600 volts can be obtained and many converters giving 1200 volts between brushes are in successful operation. In the majority of cases, however, where a line potential of 1200 volts is required two 600-volt machines are operated in series.

245. Inverted Converter. Where a small alternating-current load is to be supplied from a direct-current system, a rotary converter may be used as an inverted converter to transform direct current to alternating current.

The ratios of the voltages are the same as under normal operating conditions but the ratios of the currents vary since it is not possible to eliminate or control the wattless components of the alternating current. These components depend on the character of the load and are not affected by varying the exciting current.

When changing from alternating current to direct current the

speed of the converter is fixed by the frequency of the system and remains constant. When changing from direct current to alternating current the speed is not fixed but depends on the excitation and varies as the field strength varies. When the load is inductive the wattless lagging current demagnetizes the field and so raises the speed of the converter and the frequency of the alternating current. This may increase the lagging current and so raise the speed more until it gets beyond safe limits. The inverted converter has thus the two disadvantages, (1) that it tends to run at dangerous speeds and (2) that it supplies a current of varying frequency. It must, therefore, be provided with some means of cutting off the load when the speed rises above a certain value or with some means of limiting the speed.

If the converter is excited by a direct-current generator mounted on the same shaft any increase in speed raises the exciter voltage at a higher rate and, therefore, the field of the converter is strengthened and the speed is limited.

Rotary converters to be used as inverted converters should not be compound wound.

246. Double-current Generator. If mechanical power is supplied to drive a rotary converter it can be used as a double-current generator to supply direct current from the commutator and alternating current from the slip rings.

The two currents in this case flow in the same direction in the armature conductors and the losses are increased and the armature reaction is the sum of the reactions due to the two currents. The voltage regulation is, therefore, poorer than in the converter.

247. Three-wire Generator. The three-wire direct-current generator is similar in construction to a single-phase or quarter-phase rotary converter. It is used to supply a three-wire system with from 220 to 280 volts between outer wires and from 110 to 140 volts between each of the outer wires and the neutral wire.

In order to obtain a point at a potential midway between the potentials of the direct-current brushes special transformers called compensators are used. They have a single winding tapped at the centre and are connected by means of slip rings across points on the armature winding 180 electrical degrees apart. (Fig. 370.) The neutral wire of the system is connected to the central point of the compensator and its potential is maintained almost midway between the potentials of the outer wires. When more than one

compensator is used the centre points of all the compensators are joined together before being connected to the neutral wire. (Fig. 371.)

In some cases the compensators are connected directly to the armature windings and rotate with it and their neutral points are connected and brought out to a single slip ring.

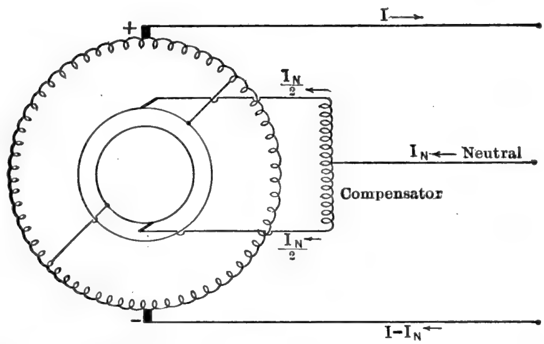


FIG. 370. Three-wire generator with a single compensator.

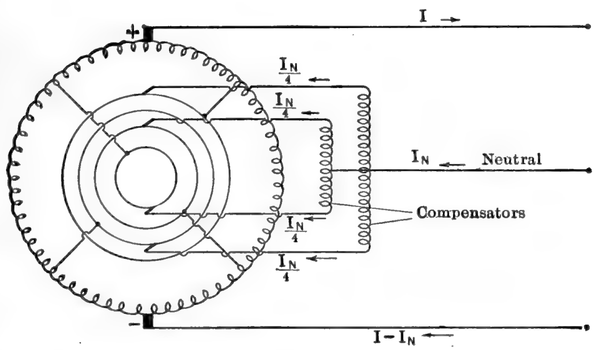


FIG. 371. Three-wire generator with two compensators.

The voltage between the terminals of the compensators is alternating and when the loads on the two sides of the three-wire system are equal and no current flows in the neutral wire the only current in the compensator is the very small exciting current.

When the loads are unequal the unbalanced current I_N flows in the neutral wire as shown in Fig. 370. On reaching the compensator it divides into two parts which flow through the winding and

up to the positive brush by the path of least resistance. Since the current in the neutral wire is a direct current the reactance of the compensator does not oppose it and the only voltage drop is that due to resistance.

The actual amount of current carried by the various sections of the armature winding is very difficult to calculate and it varies from instant to instant due to the change in the relative positions of the direct-current brushes and the compensator connections. The average current carried by each half of the compensator winding is $\frac{I_N}{2}$.

The unbalancing of the currents in the sections of the armature winding produces an unbalancing of the armature reactions and results in a slight unbalancing of the voltages between the neutral point and the brushes. With two or more compensators, Fig. 371, this unbalancing is reduced due to the more even distribution of the current.

Machines can be designed to give a regulation of 2 per cent or less with an unbalanced load of 25 per cent.

With this system the voltages on the two sides cannot be regulated independently and the flexibility of the three-wire system supplied by two generators in series is lost, but there is a corresponding gain in space and cost of machines.

The capacity required in the compensators is small. With 25 per cent unbalancing of the loads the required compensator capacity is less than 10 per cent of the generator capacity.

248. Frequency Converters. Frequency converters are used where power is transmitted at 25 cycles and is required by the consumer at 60 cycles. The most usual form of frequency converter is a synchronous motor-generator set, a 25-cycle motor direct connected to a 60-cycle alternator. The numbers of poles on the two machines must be in the ratio of 25 to 60. When a 10-pole motor is used the alternator must have 24 poles and the speed is fixed at 300 r. p. m.

When frequency converters are to be operated in parallel they must be synchronized on both the 25-cycle and 60-cycle ends. If the motor is synchronized first there is only one chance in five that the alternator is in synchronism, while if the alternator is synchronized first there is only one chance in twelve that the motor is in synchronism.

If the motor is synchronized and it is found that the generator is out of synchronism the circuit must be opened and the motor allowed to slip back a pair of poles at a time until the correct position is reached.

249. Induction Frequency Converter. The induction frequency converter may be used instead of the synchronous-motor generator set. It consists of an induction motor with a wound rotor driven by a synchronous motor connected to the 25-cycle supply lines. The stator of the induction motor is also connected to the supply and produces a revolving field. At standstill the frequency of the e.m.fs. generated in the rotor windings is 25 cycles. When the rotor is driven backwards at synchronous speed the frequency is 50 cycles and when driven at 140 per cent of synchronous speed it is 60 cycles. If a receiver circuit is connected to the rotor slip rings 60-cycle power can be supplied to it. $\frac{25}{60}$ or $\frac{5}{12}$ of the power output is supplied to the rotor by transformer action from the stator and the remaining $\frac{7}{12}$ is supplied by the synchronous motor as mechanical power.

The principal disadvantage of the induction frequency converter is its poor voltage regulation. Due to the presence of the air gap in the magnetic circuit the reactances are large and the e.m.fs. consumed by the reactances are large.

The exciting current required by the induction motor may be provided by over-exciting the fields of the synchronous motor and making it draw a leading current. In this way the power factor of the set may be made unity, but the increased current in the synchronous-motor windings increases the copper losses and the heating.

Since the rotor of an induction motor may be wound for any number of phases irrespective of the number of stator phases the induction frequency converter may be used to change the number of phases as well as the frequency. The same result may of course be obtained with the synchronous-motor generator set.

250. Mercury Arc Rectifier. The mercury arc rectifier or mercury vapor converter is used to convert alternating current to direct current for charging storage batteries supplying arc lights and many other purposes.

Fig. 372 shows the diagram of connections of a rectifier set. The essential parts are the exhausted bulb B and the two reactance coils E and F called sustaining coils. In addition there are various controlling reactances and resistances.

The bulb B has two projections on its sides containing the positive terminals or anodes A and A' which are made of graphite, and two projections on the bottom containing mercury, C is the negative terminal or cathode and S is a third anode used only for starting. The large upper space in the bulb is the cooling chamber in which the mercury vapor, which has been heated by the passage of electricity, is condensed and from which it falls down into the cathode again.

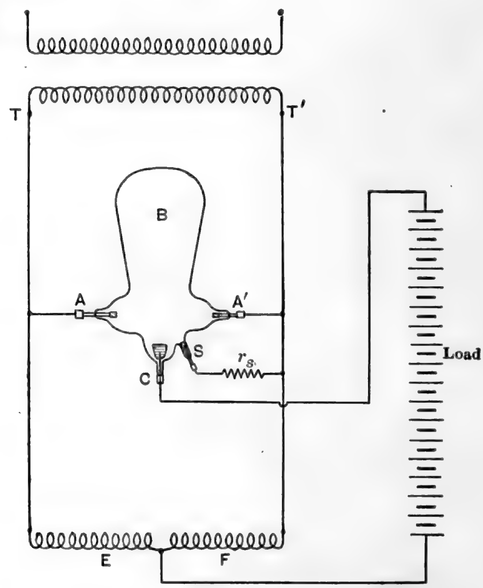


FIG. 372. Mercury arc rectifier.

The two anodes are connected to the terminals of the supply transformer TT' and the load circuit is connected between the cathode C and the junction of the two sustaining coils E and F.

251. Operation. The operation of the rectifier depends on the fact that current can pass through the tube in one direction only from either of the anodes to the cathode and it can only pass in this direction after an arc has been formed at the cathode in such a direction as to make the mercury negative. In the opposite direction the tube is a very good insulator and a difference of potential of thousands of volts would be required to produce a current. The starting arc is produced by impressing a voltage

between the two mercury terminals C and S through the starting resistance r_S and tipping the tube until the mercury forms a bridge and closes the circuit. Current then passes and when the circuit is opened by raising the tube to the vertical position an arc is formed and the cathode is said to be excited. If at this instant either of the anodes is at a higher potential than the cathode, current will flow from it and will continue to flow so long as the difference of potential is greater than the counter e.m.f. of the rectifier. When the terminal T of the supply transformer is positive, current flows from it to A through the tube C, through the load circuit and through the reactance coil F to the terminal T'. When the voltage reverses and T becomes negative, T' becomes positive and current flows from it to A' through the tube to C and through the coil E to the terminal T.

If there was no drop of voltage in the rectifier the current wave would be of the same shape as the voltage wave and in the load circuit it would vary from zero to a positive maximum as shown in Fig. 373. There is, however, a drop of voltage of from

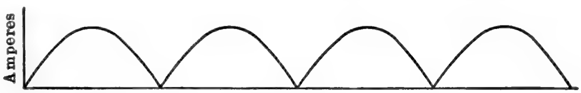


Fig. 373. Rectified alternating current.

14 volts to 25 volts in the commercial rectifier, which remains approximately constant independent of the load, and if the sustaining coils were left out of the circuit the current through the bulb would drop to zero as soon as the potential of T had fallen below the counter e.m.f. of the converter and load circuit and would remain at zero until the potential of T' rose to a value greater than this counter e.m.f. In the meantime the cathode would have lost its excitation and the tube would have to be tipped again before any current could flow. To prevent the current in the tube from falling to zero and so to insure continuous operation the reactance coils E and F are introduced. Their action is as follows: While current is flowing from T energy is stored in the magnetic field of the coil F and when the potential of T becomes too low to force the current against the counter e.m.f. of the converter the sustaining coil discharges its stored energy and maintains the current until the potential of T' rises and current flows from it to the load. The effect of the sustaining coils is there-

fore to spread out the two halves of the current wave so that they overlap and produce in the load circuit a direct current with only a slight pulsation.

252. Currents and Voltages. The voltage is controlled by a regulating reactance connected in the alternating-current supply circuit and in the ordinary rectifiers the direct voltage ranges from 20 per cent to 52 per cent of the alternating voltage while the alternating current ranges from 40 per cent to 66 per cent of the direct current.

Rectifiers are designed for direct currents of 10, 20, 30 or 40 amperes and can be built to operate on any required voltage and any frequency.

253. Losses and Efficiency. Since the counter e.m.f. of the rectifier is approximately constant independent of the load, the power losses vary directly as the current instead of as the square of the current. Neglecting the losses in the reactance coils and regulator the efficiency of the rectifier is constant at all loads and is higher the higher the voltage. Values up to 80 per cent are reached with rectifiers supplied from a 220-volt alternating-current circuit and delivering 110 volts direct current.

The power factor of the rectifier is high and under ordinary conditions may be assumed as about 90 per cent.

CHAPTER X

TRANSMISSION LINE

254. Transmission Line. The transmission line carries the electrical energy from the generating station to the receiving station or substation, where it is either transformed into mechanical energy or distributed to the customers throughout the district.

The most important characteristics of a transmission line are (1) reliability, (2) regulation and (3) efficiency.

(1) To insure reliability of service lines should, wherever possible, be installed in duplicate and all the necessary protective devices applied.

(2) For good regulation the reactance of the line should be as small as possible and therefore the frequency should be low. The capacity of a line draws a leading current, which partially counteracts the drop in voltage due to reactance and so improves the regulation.

(3) The power losses in a line are the resistance loss, which varies as the square of the current, and the comparatively small losses due to leakage over the insulators and to the formation of corona around the conductors.

To reduce the power loss the resistance of the line should be made as low as possible. This can be done by increasing the cross section of the conductors, but the increased cost of the material required soon overcomes the saving due to the increase in efficiency

For a given loss and a given voltage between lines power can be transmitted with a smaller amount of conducting material three-phase than either single-phase or two-phase.

255. Relative Amounts of Conducting Material for Single-, Two- and Three-phase Transmission Lines.

Let P = power input to the line in watts.

p = per cent loss of power in the line resistance due to full-load current.

I = full-load current.

$\cos\theta$ = power factor.

r = resistance of each conductor.

n = number of conductors in the system.

The loss in the line is

$$\frac{pP}{100} = nI^2r,$$

and the resistance of each conductor is

$$r = \frac{pP}{100\,nI^2}.$$

For the same voltage E between conductors the current is

$$I = \frac{P}{E\cos\theta}, \text{ single phase,}$$

$$I = \frac{P}{2\,E\cos\theta}, \text{ two phase,}$$

$$I = \frac{P}{\sqrt{3}\,E\cos\theta}, \text{ three phase,}$$

and the resistance per conductor is

$$r = \frac{pP \times E^2\cos^2\theta}{100 \times 2\,P^2} = 0.005\frac{pE^2\cos^2\theta}{P}, \text{ single phase,}$$

$$r = \frac{pP \times 4\,E^2\cos^2\theta}{100 \times 4\,P^2} = 0.01\frac{pE^2\cos^2\theta}{P}, \text{ two phase,}$$

$$r = \frac{pP \times 3\,E^2\cos^2\theta}{100 \times 3\,P^2} = 0.01\frac{pE^2\cos^2\theta}{P}, \text{ three phase.}$$

Since the single-phase line has only two conductors while the two-phase line has four the amount of copper required for both is the same. The three-phase line consists of three conductors of the same section as the two-phase conductors and, therefore, the amount of copper required for a three-phase line is only 75 per cent of that required for a two-phase or single-phase line with the same per cent power loss and the same maximum voltage between lines.

256. Reactance. The inductance of a line per mile of conductor is

$$L = \left(0.074 + 0.0805\log_{10}\frac{D - R}{R}\right)10^{-3} \text{ henrys,}$$

where D is the distance between conductors
and R is the radius of the conductors.

The reactance of the line per mile of conductors is

$$X = 2\,\pi f L \text{ ohms.}$$

The reactance could be decreased by decreasing the distance between the conductors or by increasing the radius, but these

quantities are fixed by other considerations than the reactance and reactance drop.

257. Capacity. The capacity of a line per mile of conductor between the conductor and neutral is

$$C = \frac{38.8}{\log_{10}\dfrac{D-R}{R}} 10^{-9} \text{ farads (Equation 45).}$$

This value applies for each conductor of a single-phase or polyphase line. If the conductors of a three-phase line are suspended in one plane instead of in the form of an equilateral triangle the capacity of the central conductor is slightly greater than that of the others, but since all lines are transposed the total capacity of each of the three is the same and is given with sufficient accuracy by the formula above if the distance D is taken as the shortest distance between conductors.

The capacity reactance per mile of conductor is

$$X_c = \frac{1}{2\,\pi f C} \text{ ohms,}$$

and the charging current per mile of conductor is

$$I_c = \frac{e}{X_c} = 2\,\pi f C e,$$

where e is the voltage between conductor and neutral.

For transmission lines up to 50,000 volts the capacity is very small and its effect on the regulation may be neglected. If, however, any part of the transmission is carried out through underground cables, the capacity may be very largely increased and may not be negligible. Above 50,000 volts the capacity of the line must be considered in calculating the regulation. For lines up to 100 miles in length and for voltages up to 100,000 volts the capacity of each conductor may be considered as a condenser connected at the centre of the line between conductor and neutral. If more accurate results are necessary the fact that both the reactance and the capacity of the line are distributed over the whole length must be taken into account.

Due to the presence of the charging current in a line the current flowing into the receiving circuit may be very much larger than the current entering the line at the generating station.

258. Voltage and Frequency. Voltages up to 110,000 volts are now in use for the transmission of large amounts of power over long distances.

Power is usually generated and transmitted at either 25 cycles or 60 cycles. With 25 cycles the reactance drop in the line is less than with 60 cycles and therefore the voltage regulation is better. In the case of very long high-voltage lines the increased charging current at the higher frequency may counteract the larger reactance drop of voltage. Where power is required for lighting 60 cycles is necessary unless frequency chargers are installed.

259. Spacing of Conductors. The distance between the conductors of a transmission line depends both on the voltage and also on certain points in the mechanical design, such as the material of the conductor, length of span and the amount of sag allowed. The curve in Fig. 374 gives approximately the relation between the spacing of the conductors and the voltage.

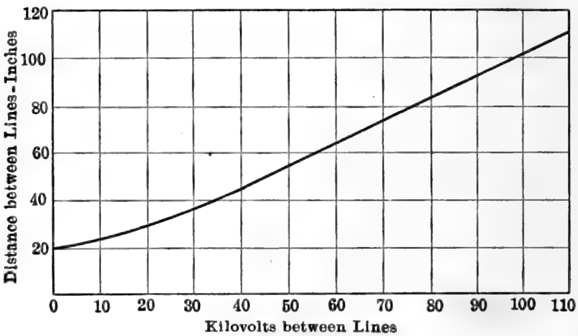


FIG. 374. Standard spacings.

260. Single-phase Transmission Line. (1) A single-phase transmission line, Fig. 375, delivers 5100 kilowatts to a receiver circuit at 60,000 volts. If the power factor of the load is 85 per cent, find the generator voltage.

r = resistance of the line = 20 ohms.

x = reactance of the line = 50 ohms.

The power delivered to the receiver circuit is

$$P = EI\cos\phi = 5{,}100{,}000 \text{ watts},$$

where $E = 60{,}000$ is the receiver voltage

and $\cos\phi = 0.85$ is the power factor;

the current is therefore

$$I = \frac{P}{E \cos \phi} = \frac{5,100,000}{60,000 \times 0.85} = 100 \text{ amperes.}$$

The vector diagram is drawn with the current $OI = I$ as horizontal.

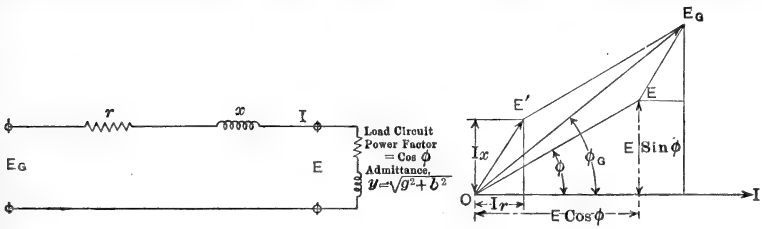


FIG. 375. Single-phase transmission line.

The receiver e.m.f. $OE = E$ leads the current by an angle ϕ and has two components

$$OE_1 = E_1 = E \cos \phi \text{ in phase with } I \text{ and}$$
$$OE_2 = E_2 = E \sin \phi \text{ in quadrature ahead of } I.$$

The voltage consumed in the resistance of the line is Ir in phase with I; the voltage consumed in the reactance of the line is Ix in quadrature ahead of I.

The component of the generator e.m.f. in phase with I is

$$E_1 + Ir = E \cos \phi + Ir$$

and the component in quadrature ahead of I is

$$E_2 + Ix = E \sin \phi + Ix,$$

and therefore the generator e.m.f. is

$$E_G = \sqrt{(E \cos \phi + Ir)^2 + (E \sin \phi + Ix)^2},$$

or substituting the numerical values

$$E_G = \sqrt{(60,000 \times 0.85 + 100 \times 20)^2 + (60,000 \times 0.52 + 100 \times 50)^2}$$
$$= 64,000 \text{ volts.}$$

The e.m.f. consumed in the line is

$$I \sqrt{r^2 + x^2} = 100 \sqrt{20^2 + 50^2} = 5400 \text{ volts.}$$

The loss of power in the line is

$$I^2 r = 100^2 \times 20^2 = 200,000 \text{ volts}$$
$$= 200 \text{ kilowatts.}$$

The power factor at the generator is

$$\cos \phi_G = \frac{E \cos \phi + Ir}{E_G} = \frac{53{,}000}{64{,}000} = 0.828 = 82.8 \text{ per cent.}$$

Using rectangular coördinates and taking the current as axis, the e.m.f. at the receiver terminals is

$$\underset{\cdot}{E} = E \cos \phi + jE \sin \phi,$$

the e.m.f. consumed in the impedance of the line is

$$\underset{\cdot}{E}' = Ir + jIx,$$

and the generator e.m.f. is

$$E_G = \underset{\cdot}{E} + \underset{\cdot}{E}' = (E \cos \phi + Ir) + j\,(E \sin \phi + Ix),$$

and its absolute value is

$$E_G = \sqrt{(E \cos \phi + Ir)^2 + (E \sin \phi + Ix)^2}.$$

The capacity of the line has been neglected in this example.

(2) A transmission line of impedance $\underset{\cdot}{z} = r + jx$ delivers power to a receiver circuit of admittance $\underset{\cdot}{y} = g - jb$ at a constant voltage E. If the capacity of the line is assumed to be concentrated at the centre determine the charging current of the line, the total current delivered by the generator and the terminal voltage of the generator.

The condensive reactance of the line is

$$x_c = \frac{1}{2\,\pi f C} \text{ ohms},$$

where f is the frequency of the impressed e.m.f. and C is the capacity of the line in farads; the condensive susceptance of the line is

$$y_c = b_c = \frac{1}{x_c}$$

and it is represented as a condenser connected across the line. (Fig. 376.)

The current in the receiver circuit is

$$\underset{\cdot}{I} = \underset{\cdot}{E}\,(g - jb),$$

and the e.m.f. at the centre of the line is

$$\begin{aligned} \underset{\cdot}{E}' &= \underset{\cdot}{E} + \underset{\cdot}{I}\left(\frac{r + jx}{2}\right) \\ &= \underset{\cdot}{E}\left\{1 + \frac{r + jx}{2}(g - jb)\right\}. \end{aligned}$$

The charging current of the line is

$$
\begin{aligned}
I_c &= E' j b_c \\
&= j b_c E \left\{ 1 + \frac{r + jx}{2} (g - jb) \right\}, \quad . \quad . \quad . \quad . \quad . \qquad (369)
\end{aligned}
$$

and the current from the generator is

$$
I_G = I + I_c = E \left[g - jb + jb_c \left\{ 1 + \frac{r + jx}{2} (g - jb) \right\} \right]. \quad (370)
$$

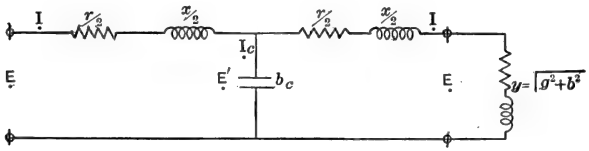


FIG. 376. Single-phase transmission line with capacity.

The terminal e.m.f. of the generator is

$$
\begin{aligned}
E_G &= E' + I_G \left(\frac{r + jx}{2} \right), \\
&= E \left\{ 1 + \frac{r + jx}{2} (g - jb) + \frac{r + jx}{2} (g - jb) + jb_c \left(\frac{r + jx}{2} \right) \right. \\
&\qquad \left. + jb_c \frac{(r + jx)^2 (g - jb)}{4} \right\} \\
&= E \left\{ 1 + (r + jx) \left(g - jb + j \frac{b_c}{2} \right) + j \frac{b_c}{4} (r + jx)^2 (g - jb) \right\}. \quad (371)
\end{aligned}
$$

For lines of small capacity the last term may be neglected and equation 371 reduces to

$$
E_G = E \left\{ 1 + (r + jx) \left(g - jb + j \frac{b_c}{2} \right) \right\}. \quad . \quad . \quad . \qquad (372)
$$

(3) A single-phase transmission line delivers 10,000 kilowatts at 100,000 volts to a receiver circuit of 85 per cent power factor; find the voltage current and power factor at the generating end of the line, the impedance drop and power loss in the line and the charging current. Find also the generator voltage required to give a receiver voltage of 100,000 volts at no load.

Length of line = 100 miles.
Size of wire = No. 000 B. & S. copper.
Diameter of wire = 2 R = 0.41 inch.
Distance between wires = D = 72 inches.
Frequency = 60 cycles per second.

The inductance or coefficient of self-induction of each wire of the line is, by equation 138,

$$L_1 = \left(0.74 \log_{10} \frac{D - R}{R} + 0.0805\right) 10^{-3} \text{ henrys per mile,}$$

and therefore the inductance of the line consisting of two wires is

$$L = 200\,L_1 = 200\left(0.74 \log_{10} \frac{71.8}{0.215} + 0.0805\right) 10^{-3} = 0.392 \text{ henrys;}$$

the inductive reactance of the line is

$$\begin{aligned} x &= 2\,\pi f L \\ &= 2 \times 3.14 \times 60 \times 0.392 = 148 \text{ ohms.} \end{aligned}$$

The capacity of each wire to neutral is, by equation 45,

$$C_1 = \frac{38.8}{\log_{10} \frac{D - R}{R}} 10^{-9} \text{ farads per mile,}$$

and the capacity between wires is

$$C_2 = \frac{C_1}{2} = \frac{19.4}{\log_{10} \frac{D - R}{R}} 10^{-9} \text{ farads per mile of line;}$$

therefore, the capacity of the line is

$$\begin{aligned} C = 100\,C_2 &= 100 \times \frac{19.4}{\log_{10} \frac{71.8}{0.215}} 10^{-9} \\ &= 0.76\ 10^{-6} \text{ farads.} \end{aligned}$$

The condensive reactance is

$$\begin{aligned} x_c &= \frac{1}{2\,\pi f C} \\ &= \frac{1}{2 \times 3.14 \times 60 \times 0.76 \times 10^{-6}} = 3480 \text{ ohms,} \end{aligned}$$

and the condensive susceptance of the line is

$$b_c = \frac{1}{x_c} = \frac{1}{3480} = 0.000287.$$

The resistance of the line at 30° C. is

$$\begin{aligned} r &= \rho \frac{l}{cir.\ mils} \text{ (Art. 56)} \\ &= 10.8 \frac{200 \times 5280}{(410)^2} = 68 \text{ ohms.} \end{aligned}$$

The load delivered to the receiver is

$$P = EI \cos \phi = 10{,}000{,}000 \text{ watts},$$

but $\qquad E = 100{,}000 \quad \text{and} \quad \cos \phi = 0.85;$

therefore, the current in the receiver circuit is

$$I = \frac{P}{E \cos \phi} = \frac{10{,}000{,}000}{100{,}000 \times 0.85} = 117 \text{ amperes};$$

the power component is

$$I_P = I \cos \phi = 117 \times 0.85 = 100 \text{ amperes};$$

the wattless component is

$$I_W = I \sin \phi = \sqrt{I^2 - I_P{}^2} = \sqrt{117^2 - 100^2} = 60 \text{ amperes}.$$

The admittance of the receiver is

$$y = \frac{I}{E} = \frac{117}{100{,}000} = 0.00117;$$

the conductance is

$$g = \frac{I \cos \phi}{E} = \frac{100}{100{,}000} = 0.001;$$

and the susceptance is

$$b = \frac{I \sin \phi}{E} = \frac{60}{100{,}000} = 0.0006.$$

The generator voltage is, by equation 371,

$$\dot{E}_G = \dot{E} \left\{ 1 + (r + jx)\left(g - jb + j\frac{b_c}{2}\right) + j\frac{b_c}{4}(r + jx)^2 (g - jb) \right\},$$

and substituting the values obtained above

$$\dot{E}_G = \dot{E} \left\{ 1 + (68 + 148\,j)\left(0.001 - 0.0006\,j + \frac{0.000287}{2}\,j\right) \right.$$
$$\left. + \frac{0.000287}{4}\,j\,(68 + 148\,j)^2 (0.001 - 0.0006\,j) \right\},$$

and simplifying

$$\dot{E}_G = \dot{E}\,(1.1355 - 0.1162\,j),$$

and the absolute value is

$$E_G = E\sqrt{(1.1355)^2 + (0.1162)^2}$$
$$= 100{,}000 \times 1.1355 = 113{,}550 \text{ volts}.$$

The current from the generator is, by equation 370,

$$\dot{I}_G = \dot{E}\left[g - jb + jb_c\left\{1 + \frac{r + jx}{2}(g - jb)\right\}\right]$$

$$= \dot{E}\left[0.001 - 0.0006\,j + 0.000287\,j\right.$$

$$\left.\left\{1 + \frac{68 + 148\,j}{2}(0.001 - 0.0006\,j)\right\}\right]$$

$$= \dot{E}\,(0.0010148 - 0.0003\,j),$$

and its absolute value is

$$I_G = 100{,}000\sqrt{(0.0010148)^2 + (0.0003)^2} = 101.5 \text{ amperes.}$$

The charging current of the line is, by equation 369,

$$\dot{I}_c = jb_c\dot{E}\left\{1 + \frac{r + jx}{2}(g - jb)\right\}$$

$$= \dot{E}\,(0.0000148 + 0.000309\,j),$$

and its absolute value is

$$I_c = 100{,}000\sqrt{(0.0000148)^2 + (0.000309)^2} = 31 \text{ amperes}$$

$$= 25.6\% \text{ of the receiver current.}$$

At no load $E = 100{,}000$, $g = 0$, $b = 0$ and the generator voltage is

$$\dot{E}_G = \dot{E}\left\{1 + (r + jx)\left(j\frac{b_c}{2}\right)\right\}$$

$$= \dot{E}\,(0.9786 + 0.0098\,j),$$

and its absolute value is

$$E_G = 100{,}000\sqrt{(0.9786)^2 + (0.0098)^2} = 97{,}860 \text{ volts.}$$

The power factor at the generator may be found by reference to the diagram in Fig. 377.

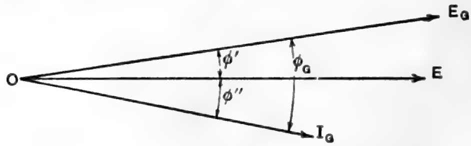


Fig. 377.

E_G leads E by an angle ϕ',
where

$$\tan\phi' = \frac{11{,}620}{113{,}550} = 0.1023;$$

I_G lags behind E by an angle ϕ'',
where

$$\tan \phi'' = \frac{3}{101.48} = 0.0296;$$

and I_G lags behind E_G by an angle $\phi_G = \phi' + \phi''$;

$$\tan \phi_G = \tan (\theta' + \theta'') = \frac{\tan \phi' + \tan \phi''}{1 - \tan \phi' \tan \phi''} = 0.132,$$

and $$\phi_G = 7° \, 30';$$

the power factor at the generator is

$$\begin{aligned} \cos \phi_G &= \cos 7° \, 30' = 0.99 \\ &= 99\%. \end{aligned}$$

The impedance drop in the line is found very approximately as

$$\begin{aligned} E_Z &= I\sqrt{r^2 + x^2} \\ &= 117 \sqrt{(68)^2 + (148)^2} = 19{,}000 \text{ volts} \\ &= 19\% \text{ of the receiver voltage.} \end{aligned}$$

The power loss in the line is found approximately as

$$\begin{aligned} P_L &= I_G'^2 \frac{r}{2} + I^2 \frac{r}{2} \\ &= (101.5)^2 34 + (117)^2 34 \\ &= 814{,}000 \text{ watts} \\ &= 814 \text{ kilowatts} \\ &= 4.07\% \text{ of the output.} \end{aligned}$$

261. Three-phase Transmission Line. A three-phase transmission line delivers 10,000 kv.a. at 60,000 volts and 60 cycles to a receiving circuit of 85 per cent power factor; determine the voltage and current at the generating station, the charging current and charging kv.a. of the line and the power loss in the line. Determine also the rise in voltage at the terminals of the receiving circuit if full load is suddenly removed.

Length of line = 77 miles.
Conductor section = 426,000 cir. mils, aluminum.
Diameter of conductor = 0.653 inch.
Distance between conductors = 72 inches.

In Fig. 378

E = voltage between lines at the receiving end = 60,000.

e = voltage between lines and neutral at the receiving end

$$= \frac{60{,}000}{\sqrt{3}}.$$

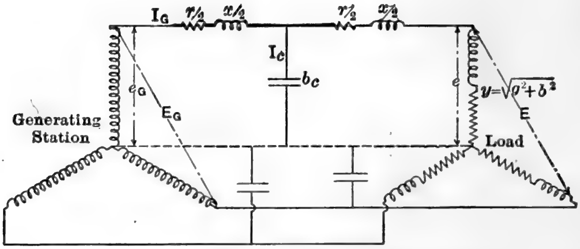


FIG. 378. Three-phase transmission line.

E_G = voltage between lines at the generating station.

e_G = voltage between lines and neutral at generating station

$$= \frac{E_G}{\sqrt{3}}.$$

I = load current in each conductor.

I_C = charging current in each conductor.

I_G = current per conductor at the generating station.

$y = \sqrt{g^2 + b^2}$ = admittance of each phase of the receiving circuit.

$z = \sqrt{r^2 + x^2}$ = impedance of each conductor.

b_c = capacity susceptance of each conductor to neutral which is assumed to be concentrated at the centre of the line.

Resistance per conductor at 20° C. is

$$r = \frac{10.35 \times 5280 \times 77}{426{,}000 \times 0.62} = 16 \text{ ohms.}$$

The conductivity of aluminum is taken as 62 per cent of that of copper.

Inductance of each conductor is

$$L = \left\{\left(0.74 \log_{10} \frac{72}{0.326} + 0.0805\right)10^{-3}\right\} 77 = 0.14 \text{ henry,}$$

and the reactance is

$$x = 2\,\pi f L = 2 \times 3.14 \times 60 \times 0.14 = 53.5 \text{ ohms.}$$

Capacity of each conductor to neutral is

$$C = \frac{38.8 \times 77}{\log_{10} \frac{72}{0.326} \times 10^9} = 1.26 \times 10^{-6} \text{ farads.}$$

The capacity reactance is

$$x_c = \frac{1}{2\pi f C} = \frac{10^6}{2 \times 3.14 \times 60 \times 1.26} = 2135 \text{ ohms,}$$

and the capacity susceptance is

$$b_c = \frac{1}{x_c} = \frac{1}{2135} = 0.000462.$$

The charging current per conductor is

$$I_c = \frac{e}{x_c} = \frac{\frac{60{,}000}{\sqrt{3}}}{2135} = 16.23 \text{ amperes,}$$

and the kv.a. required to charge the line is

$$3 \times \frac{60{,}000}{\sqrt{3}} \times \frac{16.23}{1000} = 1690.$$

The charging current and charging kv.a. have been taken as proportional to the voltage at the receiving end of the line which gives values slightly less than the true ones.

The input to the receiving circuit is 10,000 kv.a. at 85 per cent power factor $= \sqrt{3}\, EI$ and, therefore, the current per conductor is

$$I = \frac{10{,}000{,}000}{\sqrt{3} \times 60{,}000} = 92.6 \text{ amperes.}$$

The admittance of the receiving circuit per phase is

$$y = \sqrt{g^2 + b^2} = \frac{I}{e} = \frac{92.6}{\frac{60{,}000}{\sqrt{3}}} = 0.00278;$$

the conductance is

$$g = \frac{I \cos\theta}{e} = \frac{92.6 \times 0.85}{\frac{60{,}000}{\sqrt{3}}} = 0.00235$$

and the susceptance is

$$b = \sqrt{y^2 - g^2} = 0.001463.$$

The relation between the voltages at the generating station and receiving station is given by equation

$$e_G = e\left[\left\{1 + (r + jx)\left(g - jb + j\frac{b_c}{2}\right) - j\frac{b_c}{4}(r + jx)^2(g - jb)\right\}\right],$$

or

$$E_G = E\left[\left\{1 + (r + jx)\left(g - jb + j\frac{b_c}{2}\right) - j\frac{b_c}{4}(r + jx)^2(g - jb)\right\}\right].$$

Substituting the values obtained and neglecting the last term

$$E_G = 60{,}000\{1 + (16 + 53.5\,j)\,(0.00235 - 0.00123\,j)\}$$
$$= 60{,}000\,(1.1034 + 0.1063\,j),$$

and taking absolute values

$$E_G = 60{,}000\sqrt{(1.1034)^2 + (0.1063)^2} = 66{,}210 \text{ volts.}$$

Thus to produce a voltage of 60,000 volts between lines at the receiving station at full load a voltage of 66,210 volts is required at the generating station.

At no load the current I is zero and the admittance y is zero and the voltage required at the generating station to produce 60,000 volts at the receiving station is

$$E_G = 60{,}000\,\{1 + (16 + 53.5\,j)\,(-\,0.000231\,j)\}$$
$$= 60{,}000\,(0.9877 + 0.003696\,j)$$

and in absolute values

$$E_G = 60{,}000\sqrt{(0.9877)^2 + (0.003696)^2} = 59{,}268 \text{ volts.}$$

At no load therefore the voltage at the receiving end of the line is greater than that at the generating station.

If full load is suddenly taken off the line the voltage at the receiving end will rise to a value

$$E = 66{,}210 \times \frac{60{,}000}{59{,}268} = 67{,}000 \text{ volts,}$$

which is a rise of $\frac{7000}{60{,}000} \times 100 = 11.6$ per cent.

The generator voltage is here assumed to remain constant at 66,210 volts.

The current per conductor at the generating station may be found from equation 370

$$I_G = e\left[g - jb + jb_c\left\{1 + \frac{r + jx}{2}(g - jb)\right\}\right].$$

Substituting the numerical values

$$I_G = \frac{60{,}000}{\sqrt{3}}\Bigg[0.00235 - 0.00123\,j + 0.000462\,j$$

$$\left\{1 + \frac{16 + 53.5\,j}{2}(0.00235 - 0.00123\,j)\right\}\Bigg]$$

$$= \frac{60{,}000}{\sqrt{3}}(0.00229 - 0.000648\,j),$$

and taking absolute values

$$I_G = \frac{60{,}000}{\sqrt{3}}\sqrt{(0.00229)^2 + (0.000648\,j)^2} = 82.5 \text{ amperes.}$$

The power loss in the line may be taken as

$$3 \times \frac{I_G^2 + I^2}{2} \times r = 3 \times \frac{82.5^2 + 92.6^2}{2} \times 16$$

$$= 370{,}000 \text{ watts}$$

$$= 370 \text{ kilowatts.}$$

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